São Paulo Journal of Mathematical Sciences
https://doi.org/10.1007/s40863-019-00132-9

ORIGINAL ARTICLE

The *n*-dimensional Peano Curve

Joaquim E. de Freitas[1] · Ronaldo F. de Lima[1] · Daniel T. dos Santos[1]

Abstract

One of the most startling mathematical discoveries of the nineteen century was the existence of plane-filling curves. As is well known, the first example of such a curve was given by the Italian mathematician Giuseppe Peano in 1890. Subsequently, other examples of plane-filling curves appeared, with some of them having n-dimensional analogues. However, the expressions of the coordinates of the Peano curve are not easily extendable to arbitrary n dimensions. In fact, the only known extension of the Peano curve to an n-dimensional space-filling curve, made by Stephen Milne in 1982, is rather geometric and makes it difficult to establish basic properties of these curves, such as continuity and nowhere differentiability, as well as more advanced properties, such as uniform distribution of the coordinate functions. Here, we will introduce in a completely analytical way the n-dimensional version of the Peano curve. More precisely, for a given integer $n \geq 2,$ we will define (by means of identities) the n coordinate functions of a continuous and surjective map from a closed interval to the unit n-dimensional cube of $\mathbb{R}^n,$ which, for the particular case $n = 2,$ agrees with the original Peano curve. With this description, as we shall see, one can establish all the properties we mentioned above, and also calculate the Hausdorff dimension of the graphs of the coordinate functions of this curve.

Keywords Peano curve · Space-filling curve · Hausdorff dimension

✉ Ronaldo F. de Lima
ronaldo@ccet.ufrn.br

Joaquim E. de Freitas
jelias@ccet.ufrn.br

Daniel T. dos Santos
mathdaniel@hotmail.com

1 Depto. de Matemática, CCET—UFRN, Natal, Brazil

Published online: 29 April 2019

1 Introduction

At the end of the nineteenth century, mathematicians were baffled by the appearance of two kinds of continuous maps: plane-filling curves, and nowhere differentiable functions. The first example of a plane-filling curve was given by Peano [6], in 1890, who introduced a continuous and surjective map from an interval onto a square. Weierstrass, in 1872, provided the first known example of a nowhere differentiable continuous function.

Since Peano's curve became known, many mathematicians—such as Hilbert, Sierpiński, Lebesgue, and Pólya—obtained examples of plane-filling curves (see, e.g., [7]). Consequently, questions regarding the geometrical and analytical properties of these objects naturally arose. Many of these questions, in fact, remained unanswered for some time. For instance, in his paper [6], Peano announced that the coordinate functions of his curve were nowhere differentiable, but only in 1900 did Moore [5] give this statement a complete proof.

The techniques involving the construction of plane-filling curves, in general, are not easily adaptable to the n-dimensional case. For this reason, it also took some time until n-dimensional space-filling curves appeared.

It was around 1913 when Hahn and Mazurkiewicz, independently, developed a method which led to the construction of the n-dimensional version of the aforementioned Lebesgue plane-filling curve.[1] Other examples were obtained by Steinhaus [8], who proved that n-dimensional space-filling curves can be generated by stochastically independent functions. Nevertheless, none of these curves constitutes a generalisation of the Peano curve.

In [5], Moore approached the Peano curve geometrically, rather than analytically, as Peano did. By adapting Moore's methods, Milne [4] was able to construct an n-dimensional version of the Peano curve, and to prove that it is measure-preserving.

In this note, we will introduce an n-dimensional space-filling curve which is an extension of the Peano curve. It will be defined by an expression that, for the case $n = 2$, coincides with the one introduced by Peano. For this reason, it will be called the *n-dimensional Peano curve*.

We should point out that, being defined analytically, our n-dimensional version of the Peano curve is simpler than Milne's. This will allow us to establish its fundamental properties of continuity and surjectivety, as well as to characterize each of its coordinate functions as self-affine (according to Kôno [2]).

By means of this characterization and some results in [2] and [3], we will show that, in fact, these coordinate functions are $1/n$-Hölder continuous, uniformly distributed, and nowhere q-Hölder continuous for $q > 1/n$ (in particular, nowhere differentiable). We will also prove that, as a consequence of being uniformly distributed, the graphs of these functions have Hausdorff and packing dimensions both equal to $2 - 1/n$.

[1] It should me mentioned that, in contrast with the Peano curve, the Lebesgue curve is differentiable almost everywhere.

2 Definition of the *n*-dimensional Peano curve

For a given integer $n > 1$, let us denote by $[0, 1]^n$ the n-dimensional block $[0, 1] \times \cdots \times [0, 1]$ (n factors) of the Euclidean space $\mathbb{R}^n$. In what follows, we will define a continuous and surjective map $\alpha : [0, 1] \to [0, 1]^n$.

Set $D = \{0, 1, 2\}$ and let $\mathcal{S}_D$ denote the set of all sequences in D, that is

$$\mathcal{S}_D = \{(t_k)_{k\in\mathbb{N}}\, ;\; t_k \in D\}.$$

Let i be an integer such that $1 \leq i \leq n$. For each $\mathbf{t} = (t_k)_{k\in\mathbb{N}} \in \mathcal{S}_D$, consider the subsequence $\mathbf{t}_i = (t_{i+jn})_{j\geq 0}$ and define the function $x_i : \mathcal{S}_D \to [0, 1]$ by

$$x_i(\mathbf{t}) = \sum_{j=0}^{\infty} \frac{\xi^{S_{i+jn}(\mathbf{t})}(t_{i+jn})}{3^{j+1}},$$

where $\xi : D \to D$ is the operator $\xi(a) = 2 - a$, and

$$S_{i+jn}(\mathbf{t}) = \sum_{k=1}^{i+jn} t_k - \sum_{s=0}^{j} t_{i+sn}, \qquad j \geq 0. \tag{1}$$

Notice that, for fixed i and j,

(P1) $S_{i+jn}(\mathbf{t})$ depends only on the first $(i-1) + jn$ terms of $\mathbf{t}$, $t_1, \ldots, t_{(i-1)+jn}$.
(P2) $\xi^{S_{i+jn}(\mathbf{t})}(t_{i+jn})$ depends only on t_{i+jn} and the parity of $S_{i+jn}(\mathbf{t})$. More precisely, it equals t_{i+jn} if $S_{i+jn}(\mathbf{t})$ is even, and $2 - t_{i+jn}$ if $S_{i+jn}(\mathbf{t})$ is odd.

Property **(P1)** is a direct consequence of the definition of S_{i+jn}, and property **(P2)** follows from the fact that the operator ξ is an involution, that is, ξ^2 coincides with the identity map of D.

Remark 1 In many of our reasonings concerning the functions x_i, it will be convenient to represent a given $\mathbf{t} \in \mathcal{S}_D$ in the following matrix form

$$[\,\mathbf{t}\,] = \begin{bmatrix} t_1 & t_{1+n} & \ldots & t_{1+jn} & \ldots \\ \vdots & \vdots & & \vdots & \\ t_i & t_{i+n} & \ldots & t_{i+jn} & \ldots \\ \vdots & \vdots & & \vdots & \\ t_n & t_{n+n} & \ldots & t_{n+jn} & \ldots \end{bmatrix}.$$

In this way, $S_{i+jn}(\mathbf{t})$ is the sum of all entries of $[\,\mathbf{t}\,]$ from t_1 to t_{i+jn-1} [first summand in (1)], minus (if $j > 0$) the sum of the entries which are located at the i-th line, on the left of t_{i+jn} [second summand in (1)].

Define the map

$$\begin{array}{rccc} \Phi : & \mathcal{S}_D & \to & [0, 1] \\ & (t_k)_{k\in\mathbb{N}} & \mapsto & \sum \frac{t_k}{3^k} \end{array}$$

and, for $t \in [0, 1]$, call each $\mathbf{t} \in \Phi^{-1}(t)$ a *ternary representation* of t.

Let us prove that, for all $i = 1, \ldots, n$,

$$x_i(\mathbf{t}) = x_i(\mathbf{u}) \quad \forall\, \mathbf{t}, \mathbf{u} \in \Phi^{-1}(t), \;\; t \in [0, 1]. \tag{2}$$

Indeed, assuming that $\mathbf{t}$ and $\mathbf{u}$ are distinct ternary representations of $t \in [0, 1]$, we can write

- $\mathbf{t} = (t_1, \ldots, t_{(i_0-1)+j_0 n}, t_{i_0+j_0 n}, \mathbf{0})$,
- $\mathbf{u} = (t_1, \ldots, t_{(i_0-1)+j_0 n}, t_{i_0+j_0 n} - 1, \mathbf{2})$,

where $t_{i_0+j_0 n} \neq 0$ and $\mathbf{0}$, $\mathbf{2}$ denote the constant sequences equal to 0 and 2, respectively.

Since the first $(i_0 - 1) + j_0 n$ terms of $\mathbf{t}$ and $\mathbf{u}$ coincide, it follows from properties **(P1)** and **(P2)** that, for a given $i \in \{1, \ldots, n\}$, the following equality holds

$$\begin{aligned} x_i(\mathbf{t}) - x_i(\mathbf{u}) = {} & \frac{\xi^{S_{i+j_0 n}(\mathbf{t})}(t_{i+j_0 n}) - \xi^{S_{i+j_0 n}(\mathbf{u})}(u_{i+j_0 n})}{3^{j_0+1}} \\ & + \sum_{j=j_0+1}^{\infty} \frac{\xi^{S_{i+jn}(\mathbf{t})}(0) - \xi^{S_{i+jn}(\mathbf{u})}(2)}{3^{j+1}}. \end{aligned} \tag{3}$$

By considering the matrices of $\mathbf{t}$ and $\mathbf{u}$, one easily concludes that $S_{i+jn}(\mathbf{t})$ and $S_{i+jn}(\mathbf{u})$ have distinct parities in any of the following cases:

- $i < i_0$ and $j > j_0$.
- $i > i_0$ and $j \geq j_0$.

In particular, for all $i > i_0$, one has

$$\xi^{S_{i+j_0 n}(\mathbf{t})}(t_{i+j_0 n}) - \xi^{S_{i+j_0 n}(\mathbf{u})}(u_{i+j_0 n}) = \xi^{S_{i+j_0 n}(\mathbf{t})}(0) - \xi^{S_{i+j_0 n}(\mathbf{u})}(2) = 0.$$

Also, from properties **(P1)** and **(P2)**,

$$\xi^{S_{i+j_0 n}(\mathbf{t})}(t_{i+j_0 n}) - \xi^{S_{i+j_0 n}(\mathbf{u})}(u_{i+j_0 n}) = 0 \quad \forall i < i_0.$$

These facts, together with Eq. (3), give that

$$x_i(\mathbf{t}) = x_i(\mathbf{u}) \; \forall i \neq i_0.$$

Considering again the matrices of $\mathbf{t}$ and $\mathbf{u}$, one sees that, for all $j > j_0$:

- $S_{i_0+jn}(\mathbf{t}) = S_{i_0+j_0 n}(\mathbf{t}) = S_{i_0+j_0 n}(\mathbf{u})$.
- $S_{i_0+jn}(\mathbf{t})$ and $S_{i_0+jn}(\mathbf{u})$ have the same parity.

However, $t_{i_0+j_0 n} - u_{i_0+j_0 n} = 1$. Thus,

$$x_{i_0}(\mathbf{t}) - x_{i_0}(\mathbf{u}) = \frac{1}{3^{j_0+1}} - \sum_{j=j_0+1}^{\infty} \frac{2}{3^{j+1}} = 0,$$

if $S_{i_0+j_0n}(\mathbf{t})$ is even, and

$$x_{i_0}(\mathbf{t}) - x_{i_0}(\mathbf{u}) = -\frac{1}{3^{j_0+1}} + \sum_{j=j_0+1}^{\infty} \frac{2}{3^{j+1}} = 0,$$

if $S_{i_0+j_0n}(\mathbf{t})$ is odd, which implies

$$x_{i_0}(\mathbf{t}) = x_{i_0}(\mathbf{u})$$

and completes the proof of (2).

It follows from equality (2) that, for $i = 1, \dots, n,$ the functions

$$\begin{array}{rccc} x_i : & [0,1] & \to & [0,1] \\ & t & \mapsto & x_i(\mathbf{t}) \end{array},$$

$\mathbf{t} \in \Phi^{-1}(t),$ are well defined. Through them, we will introduce in the next theorem our intended n-dimensional space-filling curve.

Theorem 1 *The map*

$$\begin{array}{rccc} \alpha : & [0,1] & \to & [0,1]^n \\ & t & \mapsto & (x_1(t), \dots, x_n(t)) \end{array} \tag{4}$$

is continuous and surjective.

Proof Let $t \in [0,1]$. Given $i \in \{1, \dots, n\}$ and $k \in \mathbb{N},$ choose a ternary representation $\mathbf{t} = (t_s)_{s\in\mathbb{N}} \in \mathcal{S}_D$ of $t,$ in such a way that

$$t(k) := \sum_{s=1}^{i+kn} \frac{t_s}{3^s} + \sum_{s=(i+1)+kn}^{\infty} \frac{2}{3^s} > t.$$

It is easily seen that, if $t \le u < t(k)$ and $\mathbf{u} = (u_s)_{s\in\mathbb{N}}$ is a ternary representation of $u,$ then the first $i + kn$ terms of $\mathbf{t}$ and $\mathbf{u}$ coincide. Therefore,

$$\begin{aligned} |x_i(t) - x_i(u)| &\le \sum_{j=k+1}^{\infty} \frac{|\xi^{S_{i+jn}(\mathbf{t})}(t_{i+jn}) - \xi^{S_{i+jn}(\mathbf{u})}(u_{i+jn})|}{3^{j+1}} \\ &\le \sum_{j=k+1}^{\infty} \frac{2}{3^{j+1}} = \frac{1}{3^{k+1}}, \end{aligned}$$

which implies that x_i is continuous from the right at t.

An analogous reasoning leads to the conclusion that x_i is also continuous from the left. Thus, each coordinate function of α is continuous, which implies that the map α itself is continuous.

Now, we shall prove that α is surjective, that is, for a given point $(x_1, \ldots, x_n)$ in $[0, 1]^n$, we will obtain $t \in [0, 1]$ such that

$$x_i(t) = x_i \quad \forall i = 1, \ldots, n. \tag{5}$$

Given $i \in \{1, \ldots, n\}$, let $\mathbf{a}_i = (a_{i+jn})_{j \geq 0} \in \mathcal{S}_D$ be a ternary representation of x_i. Set $t_1 = a_1$ and, using induction, define for all $i = 1, \ldots, n$ the sequence $\mathbf{t}_i = (t_{i+jn})_{j \geq 0} \in \mathcal{S}_D$ by the equality

$$t_{i+jn} = \xi^{S_{i+jn}(\mathbf{t}_{ij})}(a_{i+jn}),$$

where $\mathbf{t}_{ij} = (t_1, \ldots, t_{(i-1)+jn}, \mathbf{0})$. It follows from property **(P1)** that

$$S_{i+jn}(\mathbf{t}) = S_{i+jn}(\mathbf{t}_{ij}) \quad \forall i = 1, \ldots, n, \quad j \geq 0.$$

Therefore, for all $i \in \{1, \ldots, n\}$ and $j \geq 0$, one has

$$\xi^{S_{i+jn}(\mathbf{t})}(t_{i+jn}) = \xi^{S_{i+jn}(\mathbf{t})}(\xi^{S_{i+jn}(\mathbf{t}_{ij})}(a_{i+jn})) = \xi^{2S_{i+jn}(\mathbf{t})}(a_{i+jn}) = a_{i+jn},$$

which implies that $t = \Phi(\mathbf{t})$ satisfies (5) and, so, that α is surjective. □

The map α defined in (4) will be called the *n-dimensional Peano curve*. For $n = 2$, it is precisely the plane-filling curve introduced by Peano in [6]. In this case, for $i = 1, 2$ and $j > 0$, the functions S_{i+jn} are simply:

- $S_{1+2j}(\mathbf{t}) = t_2 + \cdots + t_{2j}$.
- $S_{2+2j}(\mathbf{t}) = t_1 + \cdots + t_{2j+1}$.

So, regarding the construction of the n-dimensional Peano curve, our task consisted in finding suitable functions S_{i+jn}, which would generalize the above S_{1+2j} and S_{2+2j}.

3 Properties of the coordinate functions of α

We now proceed to establish the properties of the coordinate functions of the n-dimensional Peano curve mentioned at the end of the introduction. They will be derived from the main results of [2] and [3], and Propositions 1 and 2 below.

Proposition 1 *Given $k \in \mathbb{N}$ and $1 \leq i \leq n$, the i-th coordinate function x_i of the n-dimensional Peano curve α satisfies the following relation:*

$$3^k(x_i(t) - x_i(t(k))) = (-1)^{\sigma(k)} x_i(3^{kn}(t - t(k))), \tag{6}$$

where $t = \Phi(t_s)_{s \in \mathbb{N}} \in [0, 1]$, $t(k) = \Phi(t_1, \ldots, t_{kn}, \mathbf{0})$, and $\sigma(k)$ is a nonnegative integer depending on k.

Proof Writing $\mathbf{t} = (t_s)_{s\in\mathbb{N}}$, $\mathbf{t}(k) = (t_1, \dots, t_{kn}, \mathbf{0})$, and $\mathbf{u} = (t_{s+kn})_{s\in\mathbb{N}}$, we observe that

$$u := 3^{kn}(t - t(k)) = \Phi(\mathbf{u}).$$

Now, if we set

$$\sigma(k) := \sum_{q=1}^{kn} t_q - \sum_{r=0}^{k-1} t_{i+rn}$$

and consider the respective matrices of $\mathbf{t}$, $\mathbf{t}(k)$, and $\mathbf{u}$, we verify that, for all $i = 1, \dots, n$ and $j \geq 0$, the following equalities hold:

$$\sigma(k) = S_{i+(k+j)n}(\mathbf{t}(k)) = S_{i+(k+j)n}(\mathbf{t}) - S_{i+jn}(\mathbf{u}).$$

Thus, noticing that the first kn terms of $\mathbf{t}$ and $\mathbf{t}(k)$ coincide, we have that

$$\begin{aligned}
3^k(x_i(t) - x_i(t(k))) &= \sum_{j=k}^{\infty} \frac{\xi^{S_{i+jn}(\mathbf{t})}(t_{i+jn})}{3^{j-k+1}} - \sum_{j=k}^{\infty} \frac{\xi^{S_{i+jn}(\mathbf{t}(k))}(0)}{3^{j-k+1}} \\
&= \sum_{j=0}^{\infty} \frac{\xi^{S_{i+(j+k)n}(\mathbf{t})}(t_{i+(j+k)n})}{3^{j+1}} - \sum_{j=0}^{\infty} \frac{\xi^{\sigma(k)}(0)}{3^{j+1}} \\
&= \sum_{j=0}^{\infty} \frac{\xi^{\sigma(k)}\xi^{S_{i+jn}(\mathbf{u})}(t_{i+(k+j)n})}{3^{j+1}} - \sum_{j=0}^{\infty} \frac{\xi^{\sigma(k)}(0)}{3^{j+1}}.
\end{aligned}$$

Therefore,

$$3^k(x_i(t) - x_i(t(k))) = \sum_{j=0}^{\infty} \frac{\xi^{S_{i+jn}(\mathbf{u})}(t_{i+(k+j)n})}{3^{j+1}} = x_i(u),$$

if $\sigma(k)$ is even, and

$$3^k(x_i(t) - x_i(t(k))) = \sum_{j=0}^{\infty} \frac{2 - \xi^{S_{i+jn}(\mathbf{u})}(t_{i+(k+j)n})}{3^{j+1}} - \sum_{j=0}^{\infty} \frac{2}{3^{j+1}} = -x_i(u),$$

if $\sigma(k)$ is odd. In any case, we have

$$3^k(x_i(t) - x_i(t(k))) = (-1)^{\sigma(k)} x_i(3^{kn}(t - t(k))),$$

as we wished to prove. □

Following N. Kôno [2,3], we say that a function $f : [0, 1] \to \mathbb{R}$ is *self-affine* with scale parameter $H > 0$ to the integer base $r \geq 4$ if, for any integers k, s satisfying $k \geq 1$ and $0 \leq s \leq r^k - 1$, and any h satisfying $0 \leq h < r^{-k}$, one has

$$f(sr^{-k} + h) - f(sr^{-k}) = \epsilon(k, s) r^{-Hk} f(r^k h),$$

where $\epsilon(k, s) \in \{-1, 1\}$.

As pointed out in [2] and [3], a self-affine function $f : [0, 1] \to \mathbb{R}$ is not necessarily continuous. However, if f is continuous with scale parameter H, then it is H-Hölder continuous, that is, there is a constant $\lambda > 0$, such that

$$|f(t) - f(t')| \leq \lambda |t - t'|^H \quad \forall t, t' \in [0, 1].$$

In this setting, if we make $r = 3^n$, and

- $t = sr^{-k} + h = \Phi(t_s)_{s \in \mathbb{N}}$,
- $t(k) = sr^{-k} = s3^{-kn} = \Phi(t_1, \dots, t_{kn}, \mathbf{0})$,

we get from Proposition 1 the following result.

Theorem 2 *Any coordinate function x_i of the n-Peano curve α is self-affine with scale parameter $H = 1/n$ to the base $r = 3^n$. In particular, x_i is $1/n$-Hölder continuous.*

In the following, we shall prove that the coordinate functions of the n-dimensional Peano curve are uniformly distributed. With this purpose, we will consider some concepts and results from [3] (see also [9] and [10]), which we will adapt to our context.

We recall that a function f defined in an interval $I \subset \mathbb{R}$ is said to be *uniformly distributed* (with respect to the Lebesgue measure μ) if, for any measurable set $A \subset \mathbb{R}$, $f^{-1}(A)$ is measurable and $\mu(f^{-1}(A)) = \mu(A)$.

Now, given a nonnegative integer k, set

$$I_k := \left[\frac{k}{3^n}, \frac{k+1}{3^n}\right).$$

Define, for $i \in \{1, \dots, n\}$ and $s \in D = \{0, 1, 2\}$,

$$Q_i(s) := \{k \in [0, 3^n) \, ; \, \exists \, t = \Phi(\mathbf{t}) \in I_k, \, \xi^{S_i(\mathbf{t})}(t_i) = s\},$$

and denote the cardinality of $Q_i(s)$ by $|Q_i(s)|$.

Proposition 2 *For all $i \in \{1, \dots, n\}$, the function $s \in D \mapsto |Q_i(s)|$ is constant.*

Proof Let us prove first that there is a bijection between $Q_i(0)$ and $Q_i(1)$. Indeed, given $k \in Q_i(0)$, let $\mathbf{t} = (t_s)_{s \in \mathbb{N}} \in \mathcal{S}_D$ be such that

$$t = \Phi(\mathbf{t}) \in I_k \quad \text{and} \quad \xi^{S_i(\mathbf{t})}(t_i) = 0.$$

Thus, $t_i = 0$ when $S_i(\mathbf{t})$ and $t_i = 2$ when $S_i(\mathbf{t})$ is odd.

Assume that $t_i = 0$ and define

$$t' = t + \frac{1}{3^i} \quad \text{and} \quad k' = k + 3^{n-i}.$$

Writing $t'_s = t_s$ for $s \neq i$, and $t'_i = 1$, it is clear that $\mathbf{t}' = (t'_s)_{s\in\mathbb{N}}$ is a ternary representation of t'. Therefore,

$$t' \in I_{k'} = [k'/3^n, (k'+1)/3^n) \text{ and } \xi^{S_i(\mathbf{t}')}(t'_i) = 1. \tag{7}$$

Moreover, since $t \in I_k$ and $t_i = 0$, one has

$$\frac{k}{3^n} \leq t < \sum_{q=1}^{i-1} \frac{2}{3^q} + \frac{1}{3^i} = 1 - \frac{1}{3^{i-1}} + \frac{1}{3^i} \cdot$$

Thus,

$$\frac{k'}{3^n} = \frac{k + 3^{n-i}}{3^n} < 1 - \frac{1}{3^{i-1}} + \frac{2}{3^i} = 1 - \frac{1}{3^i} \leq 1 - \frac{1}{3^n},$$

which implies $k' < 3^n - 1$. This, together with (7), gives that $k' \in Q_i(1)$.

If $t_i = 2$, we define

$$t' = t - \frac{1}{3^i} \quad \text{and} \quad k' = k - 3^{n-i},$$

and conclude, analogously, that $k' \in Q_i(1)$.

Now, observe that if $t = \Phi(\mathbf{t}) \in I_k$, the hypotheses $t_i = 0$ and $t_i = 2$ are mutually exclusive. So, in an obvious way, the family of intervals $\{I_k\}_{k\in Q_i(0)}$ expresses itself as a disjoint union of two of its subfamilies. Therefore, the correspondence

$$\begin{array}{ccc} Q_i(0) & \to & Q_i(1) \\ k & \mapsto & k' = k \pm \frac{1}{3^i} \end{array}$$

is clearly a bijection, where the sign + or − is taken according to the subfamily the interval I_k belongs to.

In a similar fashion, we can construct a bijection between $Q_i(2)$ and $Q_i(1)$, which implies that the function $s \in D \mapsto |Q_i(s)|$ is, in fact, constant. □

From the definition of the coordinate functions x_i, and the fact that each of them is continuous, self-affine, and satisfies $x_i(0) = 0$, $x_i(1) = 1$, one concludes that Eq. 2.1 in [9] applies and yields

$$\sum_{s=0}^{2} |Q_i(s)| = 3^n \quad \forall i = 1, \ldots, n,$$

which, together with Proposition 2, gives

$$|Q_i(s)| = 3^{n-1} \quad \forall s \in D, \quad i = 1, \ldots, n.$$

From this last equality and Theorem 3 of [3], we obtain, as intended, the following result.

Theorem 3 *Each coordinate function of the n-dimensional Peano curve α is uniformly distributed.*

Our last result concerns Hausdorff and packing dimensions of graphs. So, we will briefly introduce these concepts. For details, we refer the reader to [1].

Let X be a nonempty subset of $\mathbb{R}^n$. Given $s \geq 0$, we define the *s-dimensional Hausdorff measure of X* as

$$\mathcal{H}^s(X) := \lim_{\delta \to 0_+} \mathcal{H}^s_\delta(X),$$

where

$$\mathcal{H}^s_\delta(X) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } X_i)^s \, : \, X \subset \bigcup_{i=1}^{\infty} X_i, \text{diam } X_i < \delta \right\},$$

and diam X_i denotes the diameter of $X_i \subset \mathbb{R}^n$. The *Hausdorff dimension* of X is then defined as

$$\dim_{\mathcal{H}} X := \inf\{s \geq 0 \, ; \, \mathcal{H}^s(X) = 0\}.$$

Assume now that X is bounded and, for all $\delta > 0$, denote by $N_\delta(X)$ the smallest number of closed balls of radius δ whose union covers X. By definition, the *upper box dimension* of X is

$$\dim_{\mathcal{B}_+} X := \lim_{\delta \to 0_+} \sup \frac{N_\delta(X)}{\log(1/\delta)},$$

and the *packing dimension* of X is

$$\dim_{\mathcal{P}} X := \inf \left\{ \sup_i \dim_{\mathcal{B}_+} X_i \, ; \, X \subset \bigcup_{i=1}^{\infty} X_i \right\}.$$

It is a well known fact that, for all bounded and nonempty set $X \subset \mathbb{R}^n$,

$$\dim_{\mathcal{H}} X \leq \dim_{\mathcal{P}} X.$$

Let us quote now a result by N. Kôno, which puts together Theorems 1 and 2 in [2].

Theorem 4 *(Kôno) Let* $f : [0, 1] \to \mathbb{R}$ *be a self-affine function with scale parameter* $H \in (0, 1)$. *Then, the following hold:*

(i) *For all* $q > H$, f *is nowhere* q*-Hölder continuous.*
(ii) *The graph* G *of* f *satisfies* $\dim_{\mathcal{H}} G = \dim_{\mathcal{P}} G = 2 - H$, *provided* f *is uniformly distributed.*

Regarding item (i), we recall that a function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called q*-Hölder continuous at* $t \in I$, if there exist $C, \delta > 0$ such that, for all $t' \in I$ satisfying $|t - t'| < \delta$, the inequality $|f(t) - f(t')| \leq C|t - t'|^q$ holds.

It follows from Theorems 2 and 3 that each coordinate function of the n-dimensional Peano curve fulfills the hypotheses of Kôno's Theorem 4, which leads to our final result:

Theorem 5 *For any coordinate function* x_i *of the* n*-dimensional Peano curve* α, *the following hold:*

(i) *For all* $q > 1/n$, x_i *is nowhere* q*-Hölder continuous. In particular,* x_i *is nowhere differentiable.*
(ii) *The Hausdorff and packing dimensions of the graph of* x_i *are both equal to* $2 - 1/n$.

References

1. Falconer, K.: Fractal Geometry: Mathematical Foundations and Applications, 2nd edn. Wiley, Hoboken (2003)
2. Kôno, N.: On self-affine functions. Jpn. J. Appl. Math. **3**, 259–269 (1986)
3. Kôno, N.: On self-affine functions II. Jpn. J. Appl. Math. **5**, 441–454 (1988)
4. Milne, S.C.: Peano curves and smoothness of functions. Adv. Math. **35**, 129–157 (1980)
5. Moore, E.H.: On certain crinkly curves. Trans. Am. Math. Soc. **1**, 72–90 (1900)
6. Peano, G.: Sur une courbe qui remplit toute une aire plane. Math. Ann. **36**, 157–160 (1890)
7. Sagan, H.: Space-filling curves. Springer, Berlin (1994)
8. Steinhaus, H.: La courbe de Peano et les fonctions indépendantes. C.R. Acad. Sci. Paris **202**, 1961–1963 (1936)
9. Urbański, M.: The probability distribution and Hausdorff dimension of self-affine functions. Probab. Theory Relat. Fields **84**, 377–391 (1990)
10. Urbański, M.: The Hausdorff dimension of the graphs of continuous self-affine functions. Proc. Am. Math. Soc. **108**(4), 921–930 (1990)