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GEOMETRICAL AND ANALYTICAL PROPERTIES OF CHEBYSHEV SETS IN RIEMANNIAN MANIFOLDS

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Abstract. We discuss Chebyshev sets of Riemannian manifolds. These are closed sets characterized by the existence of a well-defined distance-realizing projection onto them. The results we establish relate analytical properties of the distance function to these sets to their geometrical properties. They are extensions of some theorems on Chebyshev sets in Euclidean space to the context of Riemannian manifolds.

1. Introduction

In this paper we study the closed subsets of Riemannian manifolds known as Chebyshev sets, which are characterized by the existence of a well-defined distance-realizing projection onto them. Our approach relates certain analytical properties of the distance function to these sets to their geometrical properties, giving particular attention to convexity.

Chebyshev sets and their relation with convexity have long been considered in the context of Banach and Hilbert spaces (see, e.g., [2]). When bringing this theory from the context of Functional Analysis to that of Riemannian manifolds, some conceptual difficulties arise. For instance, not all geodesics on Riemannian manifolds are minimizing, and minimizing geodesics (with fixed endpoints) are not necessarily unique. As a consequence, in this context we must consider not only many different concepts of convexity, but also a special kind of Chebyshev set, called simple. A simple Chebyshev set is one for which the minimizing geodesics that realize distance to it are unique. The most appropriate formulation of convexity for our purposes here is that of total convexity (as introduced by Cheeger and Gromoll [4]), which we will adopt.

In Euclidean space, a celebrated theorem by Motzkin [16] states that, for a given closed set, the conditions of being convex and Chebyshev are equivalent. Moreover, either of these conditions is equivalent to the differentiability of the distance function to this set on its complement.

By adapting a result due to Federer [8], set in Euclidean space, Kleinjohann [15] obtained a partial generalization of Motzkin's theorem to Riemannian manifolds, proving that simple Chebyshev sets of complete connected Riemannian manifolds are precisely those for which the distance function to them is differentiable on their complement. We remark that, in general, it is not possible to extend the full

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content of Motzkin's theorem to Riemannian manifolds, since in this context there exist totally convex sets which are not simple Chebyshev sets (cf. section 4).

Let us now summarize our results. First, in Theorem 1, we characterize simple Chebyshev sets as suns (see the definition in section 4) following the aforementioned theorem of Federer and Kleinjohann. We then use this result and the classical Toponogov's Comparison Theorem to prove in Theorem 2 that a simple Chebyshev set of a complete connected Riemannian manifold M of nonnegative curvature is totally convex.[1] In this same setting, we apply Theorem 1 and Theorem 2, together with a result by Cheeger and Gromoll [4], to show that a simple Chebyshev set of M with empty boundary is a submanifold of M whose normal bundle is diffeomorphic to M (Theorem 3).

Next, again by means of Theorem 1 and Toponogov's Theorem, we establish Theorem 4, our main result. Theorem 4 states that in a complete connected Riemannian manifold M of nonnegative Ricci curvature, the distance function to a simple Chebyshev set is (strongly) subharmonic. Conversely, if the sectional curvature of M is nonnegative and the distance function to a closed set $C \subset M$ is (strongly) subharmonic, then C is a simple Chebyshev set. This constitutes an extension of a theorem due to Armitage and Kuran [1], which provides the same result in Euclidean space.

A comparison between Motzkin's and Federer-Kleinjohann's Theorems, and between Armitage-Kuran's and our Theorem 4, indicates that unlike convex sets, simple Chebyshev sets of Euclidean space share their properties with those of general Riemannian manifolds.

The paper is organized as follows. In section 2, we provide some background on a few topics in Riemannian geometry that will be used subsequently. In section 3, we introduce fundamental properties of the distance function (to a set) and the concept of a Chebyshev set. In section 4, we present (together with the Federer–Kleinjohann Theorem) our results on the differentiability of the distance function, namely, Theorems 1, 2, and 3 mentioned above. Finally, in section 5, we establish Theorem 4, which, as stated earlier, pertains to the subharmonicity of the distance function.

2. Preliminaries

In this section we fix some notation and recall same basic concepts and results in Riemannian Geometry that will be used throughout the paper. For details and proofs we refer to [6] and [18].

A Riemannian manifold M will always supposed to be n-dimensional ($n \geq 2$), complete, connected, and of class C^∞. The distance between two points $p, q \in M$ will be denoted by $\rho(p,q)$.

The open geodesic ball of M with center at $p \in M$ and radius $r > 0$ will be denoted by $B_r(p)$, and its closure by $\overline{B_r(p)}$. We will write $S_r(p)$ for the corresponding sphere $\overline{B_r(p)} - B_r(p)$.

A set $C \subset M$ is said to be *strongly convex* if, for all $p, q \in C$, there is only one minimizing geodesic of M with endpoints p, q, and that geodesic is contained in C. We say that C is *totally convex* if, for all $p, q \in C$, all geodesics of M with endpoints p, q are contained in C.

[1]This result was also obtained by Kleinjohann [14] with a different proof.

It is a well-known fact that any geodesic ball in M with sufficiently small radius is strongly convex. We also remark that a subset of M containing a single point is strongly convex but not necessarily totally convex, as shown by the points of the unit sphere S^n, and the cylinder $S^1 \times \mathbb{R}$ as well.

Let T_pM denote the tangent space of M at a point p and consider the tangent bundle of M

$$TM = \bigcup_{p \in M} T_pM$$

endowed with its canonical $2n$-dimensional Riemannian manifold structure. If we denote by $\mathscr{U}$ the set of all pairs $(p,q) \in M \times M$ for which there is a unique minimizing geodesic $\gamma_{pq} : [0,1] \to M$ from $p = \gamma_{pq}(0)$ to $q = \gamma_{pq}(1)$, then the map

$$\begin{array}{rccc} \Phi: & \mathscr{U} & \to & TM \\ & (p,q) & \mapsto & \dot{\gamma}_{pq}(0) \end{array} \tag{1}$$

is known to be continuous.

Given a Riemannian submanifold S of M with empty boundary, consider the normal bundle of S

$$TS^{\perp} = \bigcup_{p \in S} T_pS^{\perp} \subset TM$$

endowed with the metric induced by the inclusion map $TS^{\perp} \hookrightarrow TM$. Recall that the restriction of the exponential map $\exp : TM \to M$ to $TS^{\perp}$, which we denote by $\exp^{\perp}$, is called the *normal exponential map* of S. Its critical points, called *focal points* of S, appear as conjugate points to points of S along geodesics emanating from them and which are normal to S.

We state now a particular case of the classical Toponogov's Comparison Theorem which will play a crucial role in the proofs of two of our theorems.

Toponogov's Theorem. *Let M be a complete Riemannian manifold with non-negative sectional curvature, and $\gamma_i : [0,a_i] \to M$ $(i = 1,2)$ two distinct normalized geodesics emanating from $p_0 = \gamma_i(0)$ to $p_i = \gamma_i(a_i)$, one of which is minimizing. Then, if $\bar{\gamma}_i : [0,a_i] \to \mathbb{R}^2$ $(i = 1,2)$ are normalized geodesics in $\mathbb{R}^2$ emanating from $\bar{p}_0 = \bar{\gamma}_i(0)$ to $\bar{p}_i = \bar{\gamma}_i(a_i)$ which satisfy $\sphericalangle(\dot{\gamma}_1(0), \dot{\gamma}_2(0)) = \sphericalangle(\dot{\bar{\gamma}}_1(0), \dot{\bar{\gamma}}_2(0))$, one has $\rho(p_1,p_2) \leq \bar{\rho}(\bar{p}_1,\bar{p}_2)$, where $\bar{\rho}$ stands for the Euclidean distance in $\mathbb{R}^2$.*

3. Distance function – Chebyshev sets

Given a subset C of a Riemannian manifold M, we define the *distance function to C* as

$$\rho_C(p) = \inf\{\rho(p,q); q \in C\}, \ p \in M.$$

For the particular case $C = \{q\}$, we write ρ_q instead of ρ_C and call ρ_q the *distance function to q*.

Proposition 1. *Let C be a closed subset of a complete Riemannian manifold M. Then, the following hold:*

- i) *for all $p \in M$, the set $\Pi_C(p) = \{q \in C\,;\, \rho_C(p) = \rho(p,q)\}$ is nonempty;*
- ii) *if $q \in \Pi_C(p)$ and $\gamma : [0,a] \to M$ is a minimizing geodesic from p to q, then $q \in \Pi_C(\gamma(t))$ for all $t \in [0,a]$;*
- iii) *the distance function to C is 1-Lipschitz continuous.*

Proof. Property (i) follows directly from the definition of ρ_C and the fact that C is a closed set.

To prove (ii), assume one had $q \notin \Pi_C(\gamma(t))$ for some $t \in (0, a)$. In this case, there would exist $q' \in C - \{q\}$ such that $\rho_C(\gamma(t)) = \rho(\gamma(t), q') < \rho(\gamma(t), q)$. This, together with the triangle inequality, would give

$$\rho(p, q') \leq \rho(p, \gamma(t)) + \rho(\gamma(t), q') < \rho(p, \gamma(t)) + \rho(\gamma(t), q) = \rho(p, q),$$

which contradicts that $q \in \Pi_C(p)$.

Finally, given $p, p' \in M$, choose $q' \in \Pi_C(p')$. Then,

$$\rho_C(p) - \rho_C(p') \leq \rho(p, q') - \rho(p', q') \leq \rho(p, p'),$$

which implies ρ_C is 1-Lipschitz continuous. □

We say that a closed set $C \subset M$ is a *Chebyshev set* of M if $\Pi_C(p)$ contains only one point $q = \pi_C(p) \in C$ for all $p \in M$. In this case, we call the map

$$\begin{array}{rccc} \pi_C : & M & \to & C \\ & p & \mapsto & \pi_C(p) \end{array}$$

the *projection* of M onto C.

Proposition 2. *The projection π_C of a complete Riemannian manifold M onto a Chebyshev set $C \subset M$ is a continuous map.*

Proof. Let $(p_k)_{k\in\mathbb{N}}$ be a convergent sequence in M whose limit point is p. Since ρ and ρ_C are continuous functions, one has

$$\rho(\pi_C(p_k), p) \leq \rho(\pi_C(p_k), p_k) + \rho(p_k, p) = \rho_C(p_k) + \rho(p_k, p) \to \rho_C(p),$$

which implies that the sequence $(\pi_C(p_k))_{k\in\mathbb{N}}$ is bounded. Then, after passing to a subsequence, we can assume $\pi_C(p_k) \to q \in C$, for C is closed. Therefore,

$$\rho(p, q) = \lim_{k\to\infty} \rho(p_k, \pi_C(p_k)) = \lim_{k\to\infty} \rho_C(p_k) = \rho_C(p),$$

which yields $q = \pi_C(p)$. Hence, π_C is continuous. □

Example 1. In [4], Cheeger and Gromoll proved that a complete and noncompact manifold M with nonnegative sectional curvature contains a compact totally geodesic submanifold S, called the soul of M, which is a Chebyshev set. Moreover, it was proved by Perelman [17] that the projection π_S of M onto the soul S is of class C^1. This result was improved by Guijarro [12], who showed that π_S is of class C^2, and finally by Wilking [19], who showed that π_S is actually of class C^∞.

We call a Chebyshev set $C \subset M$ *simple* if, for all $p \in M$, there is only one minimizing geodesic from p to $\pi_C(p)$.

Example 2. One can easily check that all parallels and meridians of $S^1 \times \mathbb{R}$ are Chebyshev sets. The parallels are all simple Chebyshev sets and none of the meridians is simple. The set which contains only the origin of $\mathbb{R}^3$ is clearly a simple Chebyshev set of the paraboloid $z = x^2 + y^2$.

4. Differentiability of the distance function

It is a well-known fact that for a given point p of a complete Riemannian manifold M, the distance function to p is differentiable of class C^∞ on the open set of points $x \neq p$ outside the cut locus of p. At such an x, one has

$$\nabla \rho_p(x) = -\frac{\Phi(x,p)}{\|\Phi(x,p)\|},$$

where $\nabla \rho_p$ stands for the gradient of ρ_p and Φ is the map (1) (cf. [18]).

In particular, if $\{p\}$ is a simple Chebyshev set, p has empty cut locus. So ρ_p is differentiable in $M - \{p\}$. Conversely, as proved by Wolter [20], if ρ_p is differentiable in $M - \{p\}$, then $\{p\}$ is simple. The following theorem generalizes this fact for general simple Chebyshev sets. It appears in [15] and is, in fact, a straightforward adaptation of Theorem 4.8 in [8], which is set in Euclidean space. For the reader's convenience, we will sketch a proof.

Federer–Kleinjohann's Theorem. *A closed subset C of a complete connected Riemannian manifold M is a simple Chebyshev set if and only if the distance function ρ_C is differentiable in $M - C$. If so, one has*

$$\nabla \rho_C(p) = -\frac{\Phi(p,\pi_C(p))}{\|\Phi(p,\pi_C(p))\|} \quad \forall p \in M - C, \tag{2}$$

and, in particular, that ρ_C is of class C^1.

Proof. Assume that ρ_C is differentiable in $M - C$. Given $p \in M - C$, let $q \in C$ be such that $\rho_C(p) = \rho(p,q)$, and $\gamma : [0, \rho_C(p)] \to M$ a normalized minimizing geodesic from p to q. By Proposition 1-(ii), for all $t \in [0, \rho_C(p)]$, one has $\rho_C(\gamma(t)) = \rho(\gamma(t), q) = \rho_C(p) - t$. So,

$$\langle \nabla \rho_C(p), \dot\gamma(0)\rangle = \lim_{t\to 0} \frac{\rho_C(\gamma(t)) - \rho_C(p)}{t} = -1, \tag{3}$$

which, together with the Cauchy-Schwarz inequality, yields $\|\nabla \rho_C(p)\| \geq 1$. Since ρ_C is 1-Lipschitz, one also has $\|\nabla \rho_C(p)\| \leq 1$. Hence, $\|\nabla \rho_C(p)\| = 1$ and, by (3), $\nabla \rho_C(p) = -\dot\gamma(0)$. Therefore, the gradient of ρ_C at p determines the geodesic γ, which implies C is a simple Chebyshev set of M and

$$\nabla \rho_C(p) = -\dot\gamma(0) = -\frac{\Phi(p,\pi_C(p))}{\|\Phi(p,\pi_C(p))\|}.$$

Let's suppose now that C is a simple Chebyshev set. Since ρ_C is 1-Lipschitz, by Rademacher's Theorem, it is differentiable almost everywhere. Moreover, for a point $p \in M - C$ at which ρ is differentiable, the argument of the precedent paragraph applies, so $\nabla \rho_C(p)$ is given by (2).

Now, the unit field

$$p \mapsto \frac{\Phi(p,\pi_C(p))}{\|\Phi(p,\pi_C(p)\|},\ p \in M - C,$$

is clearly continuous. Thus, by using local charts, we can apply a result of Federer (namely, Lemma 4.7 of [8]) and conclude that ρ_C is differentiable in $M - C$ with gradient given by (2). □

It follows from this theorem that a compact connected Riemannian manifold M has no proper simple Chebyshev sets. Otherwise, the distance function ρ_C to such a set C would attain a maximum, contradicting the fact that the gradient of ρ_C at any point of M is a unit vector.

Definition 1. We call a closed set C of a complete Riemannian manifold M a *sun* if, given $p \in M-C$, for any $q \in \Pi_C(p)$ and any normalized geodesic $\gamma : [0,+\infty) \to M$ emanating from q which contains p and is minimizing from q to p, one has

$$\rho_C(\gamma(t)) = t \quad \forall t \in [0,+\infty). \tag{4}$$

The equality (4) gives that $\gamma(0) \in \Pi_C(\gamma(t))\ \forall t \geq 0$. However, this condition does not imply (4), even assuming that C is a Chebyshev set. Indeed, let C be a meridian of the cylinder $M = S^1 \times \mathbb{R}$. As noted earlier, C is a (not simple) Chebyshev set of M. Given $q \in C$, the parallel of M containing q is (the image of) a normalized geodesic $\gamma : [0,+\infty) \to M$, $\gamma(0) = q$. Evidently, (4) does not hold for γ, and yet $q \in \Pi_C(\gamma(t))\ \forall t \geq 0$.

It should also be noticed that any geodesic γ which satisfies the conditions of Definition 1 is a *ray*, that is, all of its segments are minimizing.

As an application of Federer–Kleinjohann's Theorem, we characterize now the simple Chebyshev sets of Riemannian manifolds as suns.

Theorem 1. *Let C be a closed subset of a complete connected noncompact Riemannian manifold M. Then, C is a sun if and only if C is a simple Chebyshev set.*

Proof. Assume C is not a simple Chebyshev subset of M. Then, there exist a point $p \in M - C$, two (possibly identical) points $q_1, q_2 \in \Pi_C(p)$, and two distinct normalized geodesics $\gamma_1, \gamma_2 : [0,+\infty) \to M$ emanating from q_1 and q_2, respectively, such that γ_1 and γ_2 intersect at p, that is, $p = \gamma_1(t_0) = \gamma_2(t_0)$, $t_0 := \rho_C(p)$.

Suppose $q_1 \neq q_2$ and choose a sufficiently small t_1 such that $\rho_C(p) < t_1 < 2\rho_C(p)$, and the point $p_1 := \gamma_1(t_1)$ is in a strongly convex geodesic ball of M centered at p. In that case, if $\dot\gamma_2(t_0) = -\dot\gamma_1(t_0)$, then p_1 is a point of γ_2 between q_2 and p. So, by Proposition 1-(ii), $\rho_C(p_1) = \rho(p_1, q_2)$. Thus,

$$\rho_C(\gamma_1(t_1)) = \rho_C(p_1) = \rho(p_1, q_2) < \rho(p, q_2) = \rho(p, q_1) < t_1\,.$$

If $\dot\gamma_2(t_0) \neq -\dot\gamma_1(t_0)$, then the geodesic from q_2 to p_1 obtained from the union of the geodesic segments q_2p (of γ_2) and pp_1 (of γ_1) is broken at p_1 and has length $\rho_C(p) + (t_1 - \rho_C(p)) = t_1$. Since broken geodesics cannot be minimizing, one has $\rho(p_1, q_2) < t_1$. Hence, $\rho_C(\gamma(t_1)) = \rho_C(p_1) \leq \rho(p_1, q_2) < t_1$.

It follows that C is not a sun if $q_1 \neq q_2$. If $q_1 = q_2$, then C is not a sun either. Indeed, in that case p is in the cut locus of q_1. So, no geodesic segment of γ_1 from q_1 to a point beyond p is minimizing.

Suppose now that C is a simple Chebyshev set. Given a point p_0 in $M - C$, let $\gamma : [0,+\infty) \to M$ be the normalized geodesic emanating from $\gamma(0) = \pi_C(p_0)$ such that $p_0 = \gamma(t_0)$, i.e., $t_0 = \rho_C(p_0) > 0$. Set

$$\Omega := \{t \in (0,+\infty);\, \rho_C(\gamma(t)) = t\}$$

and observe that Ω is nonempty and closed in $(0,+\infty)$, for $t_0 \in \Omega$, and ρ_C and γ are continuous. So, it suffices to prove that Ω is also open in $(0,+\infty)$.

Let $t_1 \in \Omega$ and write $p_1 := \gamma(t_1)$. Since $\rho_C(p_1) = t_1 > 0$, we have that $p_1 \in M - C$. Hence, we can choose $r > 0$ in such a way that $B_r(p)$ is strongly convex and $\overline{B_r(p_1)} \cap C = \emptyset$. In particular, the distance function ρ_{p_1} is differentiable in $B_r(p_1) - \{p_1\}$. Now, set $a := \max \rho_C|_{S_r(p_1)} > 0$ and choose a positive b

satisfying $b < \min\{1, r^2/a^2\}$. By Federer–Kleinjohann's Theorem, the function

$$\xi(x) = \rho_{p_1}^2(x) - b\rho_C^2(x), \ \ x \in \overline{B_r(p_1)},$$

is differentiable at any point $x \in B_r(p_1)$. Furthermore, defining in $B_r(p_1) - \{p_1\}$ and $B_r(p_1)$, respectively, the unit fields

$$v(x) = \frac{\Phi(x, p_1)}{\|\Phi(x, p_1)\|} \quad \text{and} \quad w(x) = \frac{\Phi(x, \pi_C(x))}{\|\Phi(x, \pi_C(x))\|},$$

one has $\nabla\xi(p_1) = 2b\rho_C(p_1)w(p_1) \neq 0$ and

$$\nabla\xi(x) = -2(\rho_{p_1}(x)v(x) - b\rho_C(x)w(x)) \ \forall x \in B_r(p_1) - \{p_1\}.$$

Let $p_2 \in \overline{B_r(p_1)}$ be a minimum point of ξ. Since $\xi(p_1) = -b\rho_C^2(p_1) < 0$ and, for all $x \in S_r(p_1)$, $\xi(x) = r^2 - b\rho_C^2(x) \geq r^2 - ba^2 > 0$, we have that $p_2 \in B_r(p_1)$. Therefore, $\nabla\xi(p_2) = 0$, which implies $p_2 \neq p_1$ and $\rho_{p_1}(p_2)v(p_2) = b\rho_C(p_2)w(p_2)$. Since $v(p_2)$ and $w(p_2)$ are unit vectors, this last equality yields

$$v(p_2) = w(p_2) \qquad \text{and} \qquad 0 < \rho_{p_1}(p_2) = b\rho_C(p_2) < \rho_C(p_2),$$

which means that p_1 is an interior point of the (unique) minimizing normalized geodesic σ from p_2 to $\pi_C(p_2)$. This, together with Proposition 1-(ii), gives $\pi_C(p_2) = \pi_C(p_1) = \pi_C(p_0)$. Thus, the minimizing normalized geodesic $-\sigma$ from $\pi_C(p_2)$ to p_2 is a segment of γ. Therefore, for all $t \in (0, \rho_C(p_2)) \ni t_1$, one has $\rho_C(\gamma(t)) = t$, that is, $(0, \rho_C(p_2)) \subset \Omega$. It implies Ω is open in $(0, +\infty)$ and concludes the proof. □

Henceforth, we confine ourselves to the nonnegative curvature setting. First, we apply Federer–Kleinjohann's Theorem, Theorem 1, and Toponogov's Theorem to obtain the following result, which appears in [14] with a different proof.

Theorem 2. *Every simple Chebyshev set of a complete connected noncompact Riemannian manifold M of nonnegative sectional curvature is totally convex.*

Proof. Suppose that there exists a simple Chebyshev set $C \subset M$ which is not totally convex. Then, there are (possibly identical) points $q_1, q_2 \in C$, and a geodesic $\sigma : [0, a] \to M$, $\sigma(0) = q_1$, $\sigma(a) = q_2$, which is not contained in C. Except for q_1 and q_2, we can assume without loss of generality that all points of σ are in $M - C$.

By Federer–Kleinjohann's Theorem, ρ_C is differentiable in $M - C$. Then, applying Rolle's Theorem to the function $t \mapsto \rho_C(\sigma(t))$, $t \in [0, a]$, we obtain $t_0 \in (0, a)$ such that

$$\langle \nabla\rho_C(\sigma(t_0)), \dot\sigma(t_0)\rangle = 0. \tag{5}$$

Furthermore, writing $p = \sigma(t_0) \in M - C$, one has $\nabla\rho_C(p) = \dot\gamma(t_1)$, where $\gamma : [0, +\infty) \to M$ is the normalized geodesic satisfying $\gamma(0) = \pi_C(p)$ and $\gamma(t_1) = p$, $t_1 = \rho_C(p) > 0$. It follows then from (5) that the geodesic σ is orthogonal to γ at p. Therefore, since γ is unique, if we set b for the length of the segment of σ from p to q_2, we have that $b > \rho_C(p) = t_1$. So, we can choose $t_2 > t_1$ such that

$$c := t_2 - t_1 > \frac{b^2 - t_1^2}{2t_1} > 0. \tag{6}$$

Now, by Theorem 1, γ is a ray which satisfies

$$\rho_C(\gamma(t)) = t \ \ \forall t \geq 0. \tag{7}$$

In particular, the segment of γ from p to $\gamma(t_2)$ is minimizing. Hence, considering in $\mathbb{R}^2$ a right angle whose sides have lengths b and c, it follows from Toponogov's Theorem (and the Pythagorean Theorem) that $\rho^2(\gamma(t_2), q_2) \le b^2 + c^2$ (Figure 1). This last inequality, together with (6) and (7), yields

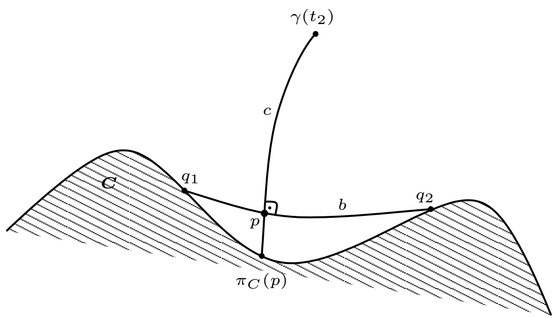

FIGURE 1

$$\rho_C^2(\gamma(t_2)) = t_2^2 = (c + t_1)^2 = c^2 + 2ct_1 + t_1^2 > c^2 + b^2 \ge \rho^2(\gamma(t_2), q_2) \ge \rho_C^2(\gamma(t_2)),$$

which, obviously, is a contradiction. □

As pointed out by Kleinjohann [14], the converse of Theorem 2 does not hold. Indeed, as showed by Gromoll and Meyer [11], there exist a manifold M of positive curvature, and a point $p \in M$ such that the set $\{p\}$ is not simple and yet is totally convex.

Given a point p in a complete Riemannian manifold M, if $\{p\}$ is a simple Chebyshev set, then $\exp_p : T_pM \to M$ is a diffeomorphism. In the next theorem, we establish that certain simple Chebyshev sets of Riemannian manifolds of nonnegative sectional curvature have a similar property. It is related to results in [13], in which the authors study the projection onto the soul (see Example 1 – section 3) of a Riemannian manifold of nonnegative sectional curvature.

Theorem 3. *Let M be a complete connected noncompact Riemannian manifold of nonnegative sectional curvature, and $C \subset M$ a simple proper connected Chebyshev set of M with empty boundary. Then, C is a totally convex submanifold of M whose normal bundle is diffeomorphic to M under the normal exponential map of C.*

Proof. As a consequence of a result established by Cheeger and Gromoll (Theorem 1.6 of [4]), any closed connected totally convex subset with empty boundary in an arbitrary Riemannian manifold is a (totally geodesic) submanifold of it. This, together with Theorem 2, gives that C is a totally convex submanifold of M.

Given $q \in C$, let $\gamma : [0, +\infty) \to M$ be a normalized geodesic emanating from q and normal to C at q, that is, $\gamma(0) = q$ and $\dot\gamma(0)$ is a unit vector in $T_qC^\perp$. Choose a point p of $\gamma - C$ sufficiently close to q in such a way that p is not conjugate to q along γ and the closed ball $\overline{B_r(p)}$ of radius $r := \rho_q(p)$ is strongly convex. In this case, the index form of $\gamma|_{[0,r]}$ is positive definite on the Jacobi fields along $\gamma|_{[0,r]}$ which are normal to $\dot\gamma$ and vanish at 0. We claim that

$$\overline{B_r(p)} \cap C = \{q\}. \tag{8}$$

Indeed, since C is totally convex and $\overline{B_r(p)}$ is strongly convex, if it were a point q_0 in $(\overline{B_r(p)} \cap C) - \{q\}$, the minimal geodesic γ_0 from q to q_0 would be entirely contained in $\overline{B_r(p)} \cap C$, which would particularly give $\langle \dot\gamma(0), \dot\gamma_0(0)\rangle = 0$. However, if we denote by σ_s the minimizing geodesic from p to $\gamma_0(s)$, $s \in [0, \rho_q(q_0)]$, a direct application of the variation formulas to the variation map $(t,s) \mapsto \sigma_s(t)$ gives that the function $s \in [0, \rho_q(q_0)] \mapsto \rho_p(\gamma_0(s))$ is strictly increasing (compare Lemma 2.3 of [4]). Thus, $r \geq \rho_p(q_0) > \rho_p(q) = r$, which is clearly a contradiction.

It follows from (8) that $q = \pi_C(p)$. Hence, by Theorem 1, γ is a ray whose points all project on q. In particular, C has no focal points in M, which implies that the normal exponential map $\exp^\perp : TC^\perp \to C$ is a local diffeomorphism. Furthermore, since C is closed, M is complete, and C is simple, we conclude that $\exp^\perp$ is bijective and, therefore, is a diffeomorphism. □

5. Subharmonicity of the distance function

We now turn our attention to the subharmonicity of the distance function ρ_C and its relation with the geometry of C. In the Euclidean case, it has been shown by Armitage and Kuran [1] that the subharmonicity of ρ_C in the complement of C is equivalent to the convexity of C.

Taking into account that the convex sets of Euclidean space are precisely its Chebyshev sets, which evidently are all simple, our next result (Theorem 4) may be considered an extension of Armitage and Kuran's Theorem to the context of Riemannian manifolds of nonnegative curvature.

Recall that a continuous function μ defined in an open set U of a Riemannian manifold M is said to be *subharmonic* if it has the following property: Given $p \in U$ and a sufficiently small $r > 0$, any harmonic function h on $B_r(p)$, continuous on $\overline{B_r(p)}$, for which $\mu \leq h$ on the boundary of $\overline{B_r(p)}$, satisfies $\mu \leq h$ in the whole of $\overline{B_r(p)}$.

We introduce now a second concept of subharmonicity by means of support functions (cf. [18]).

Definition 2. Let $\mu : U \subset M \to \mathbb{R}$ be a continuous function defined in an open subset U of a Riemannian manifold M. We say that μ is *strongly subharmonic* if, given $\epsilon > 0$, for each $p \in U$ there exist an open neighborhood V_p of p in U, and a C^∞ function $f_{p,\epsilon}$ defined in V_p which satisfies:

i) $f_{p,\epsilon}(p) = \mu(p)$;

ii) $f_{p,\epsilon}(x) \leq \mu(x)\ \forall x \in V_p$;

iii) $\Delta f_{p,\epsilon}(p) > -\epsilon$;

where $\Delta = \operatorname{div} \nabla$ stands for the Laplace operator of M. The conditions (i) and (ii) characterize $f_{p,\epsilon}$ as a *support function* for μ at p. A C^∞ function $f_{p,\epsilon}$ which fulfills the conditions (i), (ii) and (iii) above will be called an *ϵ-support function* for μ at p.

The terminology adopted in Definition 2 relies on the fact that strong subharmonicity implies subharmonicity (cf. [7, 9, 21]). Evidently, a C^∞ function whose Laplacian is nonnegative is strongly subharmonic and, hence, subharmonic.

The argument in the proof of Theorem 4 will make use of the following two lemmas. The first is attributed to Calabi [3] and appears in [7] as well.

Lemma 1 (Calabi)**.** *Let M be a complete connected Riemannian manifold of nonnegative Ricci curvature and dimension $n \geq 2$. Then, given $p \in M$, for all $x \in M$ outside the cut locus of p and different from p, one has*

$$\Delta\rho_p(x) \leq \frac{n-1}{\rho_p(x)}\,. \tag{9}$$

Lemma 2. *Let $v_1, v_2 \in \mathbb{R}^n$ be distinct vectors of $\mathbb{R}^n$ with equal lengths, that is, $\|v_1\| = \|v_2\|$. Then, there is an $\epsilon > 0$ for which the function*

$$v \mapsto \min\{\|v - v_1\|\,,\, \|v - v_2\|\},\ v \in \mathbb{R}^n,$$

does not admit any ϵ-support function at 0.

Proof. Let's write $\mu(v) = \min\{\|v - v_1\|\,,\, \|v - v_2\|\}$, $v \in \mathbb{R}^n$, and assume by contradiction that μ admits an ϵ-support function f_ϵ at 0 for all $\epsilon > 0$.

For a given $r > 0$, let $h_{\mu,r}$ be the continuous function defined in $\overline{B_r(0)}$ which is harmonic in $B_r(0)$ and coincides with μ on $S_r(0) = \partial B_r(0)$. As observed by Armitage and Kuran (see the proof of Lemma 2 of [1]), there exist positive constants a and b for which the following inequality holds:

$$\lambda(r) := \mu(0) - h_{\mu,r}(0) \geq ar - br^2. \tag{10}$$

Let $c > 0$ be such that $c < a$. By considering the limit

$$\lim_{r\to 0} \frac{ar - br^2}{e^r - 1} = a > c > 0,$$

we conclude that, for $r > 0$ sufficiently small, we have

$$\frac{\lambda(r)}{e^r - 1} > c. \tag{11}$$

Choose a positive ϵ satisfying $\epsilon < c$ and let $f = f_{\epsilon/2}$ be an $\epsilon/2$-support function for μ at 0, that is, f is a C^∞ function defined in an open set $V_0 \ni 0$ which agrees with μ at 0 and satisfies

$$\Delta f(0) \geq -\frac{\epsilon}{2} \quad \text{and} \quad f(v) \leq \mu(v)\ \forall v \in V_0\,. \tag{12}$$

For $r > 0$ sufficiently small, one has

$$ce^{-r} > \epsilon \qquad \text{and} \qquad \Delta f(v) \geq -\epsilon\ \forall v \in B_r(0). \tag{13}$$

Now, fix $r > 0$ satisfying the conditions in (11) and (13), let u be any unit vector in $\mathbb{R}^n$ and define the function

$$\xi(v) = c(e^{\langle u,v\rangle} - 1),\ v \in \overline{B_r(0)}.$$

Clearly, $\xi(0) = 0$ and ξ is of class C^∞ in $B_r(0)$. Furthermore, for all $v \in B_r(0)$, a direct calculation yields $\Delta\xi(v) = ce^{\langle u,v\rangle} > 0$, that is, ξ is subharmonic.

Let g be the function defined by

$$g(v) = f(v) + \xi(v),\ v \in \overline{B_r(0)}.$$

It follows from (13) that, for all $v \in B_r(0)$, one has

$$\Delta g(v) = \Delta f(v) + \Delta\xi(v) \geq -\epsilon + ce^{\langle u,v\rangle} \geq -\epsilon + ce^{-r} > 0,$$

which implies g is subharmonic as well.

Denote by $h_{\xi,r}$ the continuous function in $\overline{B_r(0)}$ which is harmonic in $B_r(0)$ and agrees with ξ on $S_r(0)$. Then, the function $h := h_{\mu,r} + h_{\xi,r}$ is harmonic and, for all $v \in S_r(0)$, one has

$$g(v) = f(v) + \xi(v) \leq \mu(v) + h_{\xi,r}(v) = h_{\mu,r}(v) + h_{\xi,r}(v) = h(v).$$

This, together with the subharmonicity of g, yields

$$g(v) \leq h(v) \ \ \forall v \in \overline{B_r(0)}. \tag{14}$$

On the other hand, by the maximum principle for subharmonic functions, ξ and $h_{\xi,r}$ attain their maximum on the boundary of $B_r(0)$. Hence,

$$\max h_{\xi,r} = \max \xi = c(e^r - 1).$$

Therefore, by considering (11), one has

$$g(0) = \mu(0) = \lambda(r) + h_{\mu,r}(0) > c(e^r - 1) + h_{\mu,r}(0) \geq h_{\xi,r}(0) + h_{\mu,r}(0) = h(0),$$

which contradicts (14). □

Theorem 4. *For any closed subset C of a complete connected noncompact Riemannian manifold M, one has the following:*

i) *if the Ricci curvature of M is nonnegative and C is a simple Chebyshev set, then the distance function to C is strongly subharmonic in $M - C$;*

ii) *if the curvature of M is nonnegative and the distance function to C is strongly subharmonic in $M - C$, then C is a simple Chebyshev set.*

Proof. (i) By Theorem 1, C is a sun. Thus, given $p \in M - C$, the normalized geodesic $\gamma : [0, +\infty) \to M$ which satisfies $\gamma(0) = \pi_C(p)$ and $\gamma(t_0) = p$, $t_0 = \rho_C(p) > 0$, is such that

$$\rho(\gamma(t), \pi_C(p)) = \rho_C(\gamma(t)) = t \ \ \forall t \in [0, +\infty).$$

Since γ is a ray, for a given $t > t_0$, we can choose $r = r(t) > 0$ such that $r < t - t_0$ and $B_r(p)$ is disjoint from C, from $\gamma(t)$, and from the cut locus of $\gamma(t)$. Under these assumptions, the function

$$\begin{array}{rccc} f_t: & B_r(p) & \to & \mathbb{R} \\ & x & \mapsto & t - \rho_{\gamma(t)}(x) \end{array}$$

is differentiable of class C^∞. Furthermore, f_t is a support function for ρ_C at p. Indeed, for $x = p$ one has

$$f_t(p) = t - \rho(p, \gamma(t)) = t - (t - t_0) = t_0 = \rho_C(p),$$

and, for all $x \in B_r(p)$,

$$\rho_C(x) + \rho(x, \gamma(t)) = \rho(x, \pi_C(x)) + \rho(x, \gamma(t)) \geq \rho(\gamma(t), \pi_C(x)) \geq \rho(\gamma(t), \pi_C(p)) = t.$$

Thus, for all $x \in B_r(p)$ one has $\rho_C(x) \geq t - \rho(x, \gamma(t)) = f_t(x)$, which proves our claim.

Now, considering Calabi's estimate (9) and, for $x \in B_r(p)$, the inequalities

$$\rho_{\gamma(t)}(x) \geq \rho(p, \gamma(t)) - \rho(x, p) \geq (t - t_0) - r,$$

one has

$$\Delta f_t(x) = -\Delta \rho_{\gamma(t)}(x) \geq \frac{1-n}{\rho_{\gamma(t)}(x)} \geq \frac{1-n}{t - (t_0 + r)} \ \ \forall x \in B_r(p).$$

Hence, given $\epsilon > 0$, choosing t sufficiently large, we get $\Delta f_t(p) \geq -\epsilon$, which implies f_t is an ϵ-support function for ρ_C at p. Therefore, ρ_C is strongly subharmonic in $M - C$.

(ii) Let us suppose that C is not a simple Chebyshev set and show that ρ_C is not strongly subharmonic in $M - C$. Then, we are assuming that there are points $p \in M - C$, $q_1, q_2 \in C$ (with q_1 possibly identical to q_2) and two distinct vectors $v_1\,, v_2 \in T_pM$, which satisfy

$$\exp_p(v_1) = q_1\,, \quad \exp_p(v_2) = q_2 \quad \text{and} \quad \|v_1\| = \|v_2\| = \rho_C(p). \tag{15}$$

We shall prove that there exists an $\epsilon > 0$ for which ρ_C does not admit any ϵ-support function at p, leading us to the desired conclusion. With this purpose, consider the function

$$\mu(x) = \min\{\rho(x, q_1), \rho(x, q_2)\},\ x \in M,$$

and observe that $\mu(p) = \rho_C(p)$, and $\mu(x) \geq \rho_C(x)$ for all $x \in M$. So, any ϵ-support function for ρ_C at p is an ϵ-support function for μ at p as well. Therefore, it suffices to prove that there exists an $\epsilon > 0$, for which μ does not admit any ϵ-support function at p. To accomplish that, we will assume the opposite and then derive a contradiction.

So, let's suppose that, for all given $\epsilon > 0$, there exists an ϵ-support function $f = f_{p,\epsilon} : B_p(r) \subset M \to \mathbb{R}$ for μ at p. By taking r sufficiently small, we can assume that $B_r(p)$ is strongly convex and then consider the normal coordinate system around p defined by the map

$$\begin{array}{rccc} \varphi : & B_p(r) & \to & B_0(r) \\ & x & \mapsto & \exp_p^{-1}(x), \end{array}$$

where $B_0(r)$ is the open ball of T_pM centered at the origin and of radius r. The formula for the Laplacian in coordinate systems applied to φ gives that, at p, the Laplacian of f coincides with the Euclidean Laplacian $\bar{\Delta}$ of $\bar{f} := f \circ \varphi^{-1}$ at the origin of T_pM (cf. [18]), that is,

$$\bar{\Delta}\bar{f}(0) = \Delta f(p) > -\epsilon. \tag{16}$$

Now, consider the function

$$\bar{\mu}(v) = \min\{\|v - v_1\|\,, \|v - v_2\|\},\ v \in B_0(r),$$

where v_1 and v_2 are the vectors defined in (15), and observe that

$$\bar{f}(0) = f(p) = \mu(p) = \bar{\mu}(0). \tag{17}$$

For a given $v \in B_0(r) - \{0\}$, let $q = \exp_p(v) \in B_p(r) - \{p\}$ and suppose, without loss of generality, that $\bar{\mu}(v) = \|v - v_1\|$. We have that $\|v\| = \rho(p, q)$, $\|v_1\| = \rho(p, q_1)$, and v and v_1 are tangent vectors, at p, of the geodesics from p to q and from p to q_1, respectively. Thus, we can apply Toponogov's Theorem and conclude that the inequality $\rho(q, q_1) \leq \|v - v_1\|$ holds (Figure 2). Therefore,

$$\bar{f}(v) = f(\exp_p(v)) = f(q) \leq \mu(q) \leq \rho(q, q_1) \leq \|v - v_1\| = \bar{\mu}(v). \tag{18}$$

It follows from (16), (17) and (18) that $\bar{f}$ is an ϵ-support function for $\bar{\mu}$ at 0. Since ϵ has been chosen arbitrarily, this fact contradicts Lemma 2 and, hence, concludes the proof. □

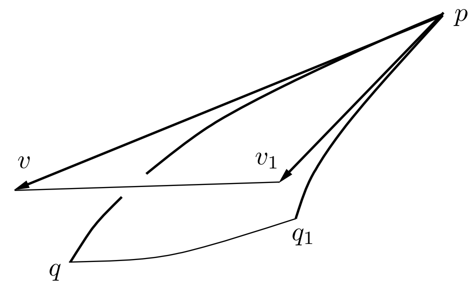

FIGURE 2

Remark 1. Since strongly subharmonicity implies subharmonicity, under the hypothesis of part (i) of the preceding theorem, we can conclude that the distance function ρ_C to a simple Chebyshev set C of M is subharmonic. An alternate proof of this fact can be performed by showing first that, for sufficiently large t, the function f_t considered in our proof is convex.[2] This, together with the support principle for convexity, as stated in Section 2 of [9], gives that ρ_C itself is convex. The subharmonicity of ρ_C follows then from Theorem 1 of [10], according to which, any convex function on a C^∞ Riemannian manifold is subharmonic. We are indebted to the referee for pointing out this proof, as well as to Arlandson Oliveira for letting us know about Greene's article [9].

From Theorem 2 and Theorem 4-(ii) we obtain the following result.

Corollary 1. *Let C be a closed subset of a complete connected noncompact Riemannian manifold M of nonnegative curvature. If the distance function to C is strongly subharmonic, then C is totally convex.*

REFERENCES

[1] D. H. Armitage and Ü. Kuran, *The convexity of a domain and the superharmonicity of the signed distance function*, Proc. Amer. Math. Soc. **93** (1985), no. 4, 598–600, DOI 10.2307/2045528. MR776186 (86k:31005)

[2] V. S. Balaganskiĭ and L. P. Vlasov, *The problem of the convexity of Chebyshev sets* (Russian), Uspekhi Mat. Nauk **51** (1996), no. 6(312), 125–188, DOI 10.1070/RM1996v051n06ABEH003002; English transl., Russian Math. Surveys **51** (1996), no. 6, 1127–1190. MR1440157 (98e:46007)

[3] E. Calabi, *An extension of E. Hopf's maximum principle with an application to Riemannian geometry*, Duke Math. J. **25** (1958), 45–56. MR0092069 (19,1056e)

[4] Jeff Cheeger and Detlef Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443. MR0309010 (46 #8121)

[5] R. F. de Lima, *Uma nova demonstração do Teorema de Motzkin*, Rev. Mat. Univ., **50–51**, 76–82 (2014).

[6] M. P. do Carmo, *Riemannian Geometry*, Birkhäuser (1993).

[7] Jost Eschenburg and Ernst Heintze, *An elementary proof of the Cheeger-Gromoll splitting theorem*, Ann. Global Anal. Geom. **2** (1984), no. 2, 141–151, DOI 10.1007/BF01876506. MR777905 (86h:53042)

[8] Herbert Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491. MR0110078 (22 #961)

[2]This is proved in Section 4 of [9] in the nonnegative sectional curvature setting. However, the same argument works assuming that M has nonnegative Ricci curvature, since it is based on Calabi's estimate (9).

[9] Robert E. Greene, *A genealogy of noncompact manifolds of nonnegative curvature: history and logic*, Comparison geometry (Berkeley, CA, 1993), Math. Sci. Res. Inst. Publ., vol. 30, Cambridge Univ. Press, Cambridge, 1997, pp. 99–134. MR1452869 (98g:53069)

[10] R. E. Greene and H. Wu, *On the subharmonicity and plurisubharmonicity of geodesically convex functions*, Indiana Univ. Math. J. **22** (1972/73), 641–653. MR0422686 (54 #10672)

[11] Detlef Gromoll and Wolfgang Meyer, *On complete open manifolds of positive curvature*, Ann. of Math. (2) **90** (1969), 75–90. MR0247590 (40 #854)

[12] Luis Guijarro, *On the metric structure of open manifolds with nonnegative curvature*, Pacific J. Math. **196** (2000), no. 2, 429–444, DOI 10.2140/pjm.2000.196.429. MR1800586 (2002b:53042)

[13] Luis Guijarro and Gerard Walschap, *The metric projection onto the soul*, Trans. Amer. Math. Soc. **352** (2000), no. 1, 55–69, DOI 10.1090/S0002-9947-99-02237-0. MR1487617 (2000c:53034)

[14] Norbert Kleinjohann, *Convexity and the unique footpoint property in Riemannian geometry*, Arch. Math. (Basel) **35** (1980), no. 6, 574–582 (1981), DOI 10.1007/BF01235383. MR604257 (82i:53047)

[15] Norbert Kleinjohann, *Nächste Punkte in der Riemannschen Geometrie* (German), Math. Z. **176** (1981), no. 3, 327–344, DOI 10.1007/BF01214610. MR610214 (82k:49053)

[16] T. Motzkin, *Sur quelques propriétés charactéristique des ensembles convexes.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **21**, 562–567 (1935).

[17] G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll*, J. Differential Geom. **40** (1994), no. 1, 209–212. MR1285534 (95d:53037)

[18] Takashi Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author. MR1390760 (97f:53001)

[19] Burkhard Wilking, *A duality theorem for Riemannian foliations in nonnegative sectional curvature*, Geom. Funct. Anal. **17** (2007), no. 4, 1297–1320, DOI 10.1007/s00039-007-0620-0. MR2373019 (2009a:53047)

[20] Franz-Erich Wolter, *Distance function and cut loci on a complete Riemannian manifold*, Arch. Math. (Basel) **32** (1979), no. 1, 92–96, DOI 10.1007/BF01238473. MR532854 (80h:53045)

[21] H. Wu, *An elementary method in the study of nonnegative curvature*, Acta Math. **142** (1979), no. 1-2, 57–78, DOI 10.1007/BF02395057. MR512212 (80c:53054)

[22] H. Wu, *On a problem concerning the intrinsic characterization of* $\mathbf{C}^n$, Math. Ann. **246** (1979/80), no. 1, 15–22, DOI 10.1007/BF01352022. MR554128 (81a:32008)

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