Results in Mathematics

Einstein Hypersurfaces of Warped Product Spaces

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Abstract. We consider Einstein hypersurfaces of warped products $I \times_\omega \mathbb{Q}^n_\epsilon$, where $I \subset \mathbb{R}$ is an open interval and $\mathbb{Q}^n_\epsilon$ is the simply connected space form of dimension $n \geq 2$ and constant sectional curvature $\epsilon \in \{-1, 0, 1\}$. We show that, for all $c \in \mathbb{R}$ (resp. $c > 0$), there exist rotational hypersurfaces of constant sectional curvature c in $I \times_\omega \mathbb{H}^n$ and $I \times_\omega \mathbb{R}^n$ (resp. $I \times_\omega \mathbb{S}^n$), provided that ω is nonconstant. We also show that the gradient T of the height function of any Einstein hypersurface of $I \times_\omega \mathbb{Q}^n_\epsilon$ (if nonzero) is one of its principal directions. Then, we consider a particular type of Einstein hypersurface of $I \times_\omega \mathbb{Q}^n_\epsilon$ with non vanishing T—which we call *ideal*—and prove that, for $n > 3$, such a hypersurface Σ has either precisely two or precisely three distinct principal curvatures everywhere. We show that, in the latter case, there exist such a Σ for certain warping functions ω, whereas in the former case Σ is necessarily of constant sectional curvature and rotational, regardless the warping function ω. We also characterize ideal Einstein hypersurfaces of $I \times_\omega \mathbb{Q}^n_\epsilon$ with no vanishing angle function as local graphs on families of isoparametric hypersurfaces of $\mathbb{Q}^n_\epsilon$.

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1. Introduction

In this paper, we consider Einstein hypersurfaces of warped products $I \times_\omega \mathbb{Q}^n_\epsilon$, where $I \subset \mathbb{R}$ is an open interval, and $\mathbb{Q}^n_\epsilon$ stands for the simply connected space form of dimension $n \geq 2$ and constant sectional curvature $\epsilon \in \{-1, 0, 1\}$. Recall that a Riemannian manifold (M, g) is called *Einstein* if

$$\mathrm{Ric}_M = \Lambda g,$$

where Ric_M is the Ricci tensor of (M, g) and Λ is a constant. More specifically, we call such an M a Λ-Einstein manifold.

Riemannian manifolds of constant sectional curvature (CSC, for short) are the simplest examples of Einstein manifolds, which we call *trivial*. It was shown by P. Ryan [14] that, for $\epsilon \leq 0$, any Einstein hypersurface of $\mathbb{Q}_\epsilon^{n+1}$ is trivial, and also that there exist nontrivial Einstein hypersurfaces in $\mathbb{S}^{n+1}$. Inspired by this result, the main question we address here is the following:

Under which conditions an Einstein hypersurface of

$I \times_\omega \mathbb{Q}_\epsilon^n$ *is necessarily trivial?*

For ω constant and $\epsilon \neq 0$, it was proved in [11] that any Einstein hypersurface of $\mathbb{R} \times_\omega \mathbb{Q}_\epsilon^n$ is trivial. On the other hand, this is not true for certain nonconstant warping functions ω. Indeed, writing $\mathbb{S}^2(1/\sqrt{2})$ for the 2-sphere of $\mathbb{R}^3$ of radius $1/\sqrt{2}$, it is known that the product $\mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$—endowed with its canonical Riemannian metric—is an Einstein manifold of nonconstant sectional curvature which is naturally embedded in $\mathbb{S}^5 - \{p, -p\} = (0, \pi) \times_{\sin t} \mathbb{S}^4$, where p is a suitable point of the unit sphere $\mathbb{S}^5 \subset \mathbb{R}^6$.

It should be noticed that, in answering the above question, one may conclude that certain manifolds cannot be isometrically immersed in $I \times_\omega \mathbb{Q}_\epsilon^n$. For instance, from the aforementioned result in [11], there is no isometric immersion of the product $\mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^2(1/\sqrt{2})$ into $\mathbb{R} \times \mathbb{Q}_\epsilon^4$, $\epsilon \neq 0$.

A major property of Einstein hypersurfaces of $I \times_\omega \mathbb{Q}_\epsilon^n$ we establish here is that they have the gradient T of their height functions (when nonzero) as principal directions. Hypersurfaces satisfying this condition, which we call the T-*property*, have been considered in several works on constant curvature (mean and sectional) hypersurfaces of product spaces $\mathbb{R} \times M$, M a Riemannian manifold (see, e.g., [5,6,11–13,15]). We add that, in product spaces $I \times_\omega \mathbb{Q}_\epsilon^n$, the class of hypersurfaces with the T-property is abundant and includes all rotational ones.

Considering our purposes here, a primary concern should be to prove existence of trivial Einstein hypersurfaces in $I \times_\omega \mathbb{Q}_\epsilon^n$. This is accomplished in our first result, as stated below. It should be mentioned that we conceive rotational hypersurfaces as in [5], dividing them into three types: spherical, parabolic, and hyperbolic (see Sect. 4 for details).

Theorem 1. *Given an open interval $I \subset \mathbb{R}$, let $\omega : I \to \omega(I)$ be a diffeomorphism such that $|\omega'| > 1$ on I. Then, the following assertions hold:*

   i) *For all $c \in \mathbb{R}$, there exists a rotational spherical-type hypersurface of constant sectional curvature c in $I \times_\omega \mathbb{Q}_\epsilon^n$, where $\epsilon \in \{0, -1\}$.*
   ii) *For all $c > 0$, there exists a rotational spherical-type hypersurface of constant sectional curvature c in $I \times_\omega \mathbb{S}^n$.*
   iii) *For all $c < 0$, there exist in $I \times_\omega \mathbb{H}^n$ a rotational parabolic-type hypersurface and a rotational hyperbolic-type hypersurface which are both of constant sectional curvature c.*

iv) *There exists a flat rotational parabolic-type hypersurface in $I \times_\omega \mathbb{H}^n$.*

We remark that Theorem 1 is of local nature. So, there is no loss of generality in assuming the (nonconstant) warping function ω to be a diffeomorphism over its image. By the same token, we can assume that, on I, $|\omega'|$ is bounded below by a positive constant. Hence, up to scaling, one has $|\omega'| > 1$. Considering these facts, Theorem 1 shows that, for virtually any nonconstant warping function ω, the class of local rotational CSC hypersurfaces of $I \times_\omega \mathbb{Q}^n_\epsilon$ is abundant.

To each warped product $I \times_\omega \mathbb{Q}^n_\epsilon$, we associate two real functions α and β defined on I, which arise from the expression of the curvature tensor of $I \times_\omega \mathbb{Q}^n_\epsilon$ in terms of the curvature tensor of $\mathbb{Q}^n_\epsilon$ and the unit tangent ∂_t to the horizontal factor $I \subset \mathbb{R}$. Then, assuming $n > 3$, we show that a CSC hypersurface Σ of $I \times_\omega \mathbb{Q}^n_\epsilon$ is essentially rotational. More precisely, we have the following result (ξ always denotes the height function of the hypersurface).

Theorem 2. *Let $\Sigma \subset I \times_\omega \mathbb{Q}^n_\epsilon$ ($n > 3$) be a hypersurface with constant sectional curvature on which $(\beta \circ \xi)T$ never vanishes. Then, Σ is rotational.*

Next, we turn our attention to Einstein hypersurfaces of $I \times_\omega \mathbb{Q}^n_\epsilon$ focusing on our main question. Firstly, as stated in the next two theorems, we characterize them according to whether $(\beta \circ \xi)T$ vanishes everywhere or nowhere on the hypersurface.

Theorem 3. *For $n > 3$, let Σ be a connected Λ-Einstein hypersurface of $I \times_\omega \mathbb{Q}^n_\epsilon$ such that $(\beta \circ \xi)T$ vanishes on Σ. Then, $\alpha \circ \xi$ is constant. Furthermore, defining*

$$\sigma := \Lambda + (n-1)(\alpha \circ \xi),$$

the following assertions hold:

i) *If $\sigma > 0$, Σ is trivial and totally umbilical (with constant umbilical function).*

ii) *If $\sigma = 0$, Σ is trivial and its shape operator has rank at most one.*

iii) *If $\sigma < 0$, Σ is nontrivial and has precisely two distinct nonzero constant principal curvatures with opposite signs, both with constant multiplicity ≥ 2. In particular, Σ has constant mean curvature.*

Connected Einstein hypersurfaces on which $(\beta \circ \xi)T$ never vanishes have many interesting properties. For this reason, we shall call them *ideal.* The following result supports our claim.

Theorem 4. *For $n > 3$, let Σ be an ideal Einstein hypersurface of $I \times_\omega \mathbb{Q}^n_\epsilon$. Then, Σ has the T-property. In addition, the following assertions are equivalent:*

i) *Σ is trivial.*

ii) *Any vertical section $\Sigma_t = \Sigma \,\overline{\pitchfork}\, (\{t\} \times \mathbb{Q}^n_\epsilon)$ is totally umbilical.*

iii) *Σ is rotational.*

The second fundamental property of ideal Einstein hypersurfaces of $I \times_\omega \mathbb{Q}_\epsilon^n$ we establish here is that they are local horizontal graphs of functions $\phi = \phi(s)$ whose level hypersurfaces f_s are parallel and isoparametric (i.e, have constant principal curvatures) in $\mathbb{Q}_\epsilon^n$. We call such a hypersurface a (ϕ, f_s)-*graph*.

Theorem 5. *For $n > 3$, let Σ be an oriented ideal Einstein hypersurface of $I \times_\omega \mathbb{Q}_\epsilon^n$ with unit normal N and non vanishing angle function $\theta := \langle N, \partial_t \rangle$. Under these conditions, the following hold:*

  i) *Σ is locally a (ϕ, f_s)-graph in $I \times_\omega \mathbb{Q}_\epsilon^n$.*
  ii) *The parallel family $\{f_s\}$ of any local (ϕ, f_s)-graph of Σ is isoparametric, and each f_s has at most two distinct principal curvatures.*
  iii) *Σ is trivial, provided that the principal curvature associated to T vanishes everywhere on Σ.*

Theorem 3-(iii) opens the possibility of existence of nontrivial Einstein hypersurfaces in $I \times_\omega \mathbb{Q}_\epsilon^n$. As we have seen, they are easily obtained in $\mathbb{S}^{n+1}$, which turns out to be a space form that can be expressed as a warped product. However, nontrivial examples of Einstein hypersurfaces of general warped products $I \times_\omega \mathbb{Q}_\epsilon^n$ are not easy to find. Here, we consider this problem restricting ourselves to the class of constant angle hypersurfaces of $I \times_\omega \mathbb{Q}_\epsilon^n$, obtaining the following result.

Theorem 6. *For $n > 3$, let Σ be a connected oriented Λ-Einstein hypersurface (not necessarily ideal) of $I \times_\omega \mathbb{Q}_\epsilon^n$ with constant angle function $\theta \in [0, 1)$. Then, the following assertions hold:*

  i) *If $\theta \neq 0$, Σ is trivial.*
  ii) *If $\theta = 0$, ω is necessarily a solution of the ODE:*

$$(\omega')^2 + \frac{\Lambda}{n-1}\omega^2 + c = 0, \ c \in \mathbb{R},$$

  on an open subinterval of I, and $I \times_\omega \mathbb{Q}_\epsilon^n$ has nonconstant sectional curvature if $c + \epsilon \neq 0$. Regarding Σ, we distinguish the following cases, according to the sign of $c + \epsilon$:
    a) *$c + \epsilon = 0 : \Sigma$ is trivial and its shape operator has rank at most one.*
    b) *$c + \epsilon < 0 : \Sigma$ is trivial and constitutes a cylinder over a totally umbilical hypersurface of $\mathbb{Q}_\epsilon^n$.*
    c) *$c + \epsilon > 0 : \Sigma$ is nontrivial and constitutes a cylinder over a product of two spheres of $\mathbb{S}^n$ (so that ϵ is necessarily 1 in this case).*

We point out that Theorem 6-(ii)(c) provides examples of nontrivial Einstein hypersurfaces in warped products $I \times_\omega \mathbb{S}^n$ which are not space forms.

Putting together Theorems 4–6 with classical results on classification of isoparametric hypersurfaces in space forms, we obtain the following characterization of ideal Einstein hypersurfaces of $I \times_\omega \mathbb{Q}_\epsilon^n$.

Theorem 7. *Let Σ be an ideal Einstein hypersurface of $I \times_\omega \mathbb{Q}_\epsilon^n$, $n > 3$. Then, Σ has a constant number of distinct principal curvatures—which is two or three— all with constant multiplicities. Moreover, Σ is trivial if and only if it has two distinct principal curvatures everywhere. If so, Σ is necessarily rotational.*

The paper is organized as follows. In Sect. 2, we set some notation and formulae. In Sect. 3, we establish three key lemmas. In Sect. 4, we consider rotational CSC hypersurfaces of $I \times_\omega \mathbb{Q}_\epsilon^n$ and prove Theorems 1 and 2. Section 5 is reserved for the proofs of Theorems 3 and 4. In Sect. 6, we introduce the (ϕ, f_s)-graphs we mentioned, establishing some of its fundamental properties. In the final Sect. 7, we prove Theorems 5–7.

Added in proof. After a preliminary version of this paper was completed, we became acquainted with the work of V. Borges and A. da Silva [2], which overlaps with ours on some parts.

2. Preliminaries

2.1. Ricci Tensor

Given a Riemannian manifold $(M^n, \langle\,,\,\rangle)$ with curvature tensor R and tangent bundle TM, recall that its *Ricci tensor* is defined as:

$$\mathrm{Ric}\,(X, Y) := \mathrm{trace}\,\{Z \mapsto R(Z, X)Y\}, \quad (X, Y) \in TM \times TM.$$

We say that a frame $\{X_1, \dots, X_n\} \subset TM$ *diagonalizes* the Ricci tensor of $(M, \langle\,,\,\rangle)$ if $\mathrm{Ric}\,(X_i, X_j) = 0 \,\forall i \neq j \in \{1, \dots, n\}$. Notice that, when M is an Einstein manifold, any orthogonal frame $\{X_1, \dots, X_n\}$ in TM diagonalizes its Ricci tensor.

2.2. Hypersurfaces

Let Σ be an oriented hypersurface of a Riemannian manifold $\overline{M}^{n+1}$ whose unit normal field we denote by N. Let A be the shape operator of Σ with respect to N, that is,

$$AX = -\overline{\nabla}_X N, \ X \in T\Sigma,$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{M}^{n+1}$.

For such a Σ, and for $X, Y, Z, W \in T\Sigma$, one has the *Gauss equation*:

$$\langle R(X, Y)Z, W \rangle = \langle \overline{R}(X, Y)Z, W \rangle + \langle AX, W \rangle \langle AY, Z \rangle - \langle AX, Z \rangle \langle AY, W \rangle, \tag{1}$$

where R and $\overline{R}$ denote the curvature tensors of Σ and $\overline{M}^{n+1}$, respectively.

Let $\lambda_1, \dots, \lambda_n$ be the principal curvatures of Σ, and $\{X_1, \dots, X_n\}$ the corresponding orthonormal frame of principal directions, i.e.,

$$AX_i = \lambda_i X_i, \quad \langle X_i, X_j \rangle = \delta_{ij}.$$

In this setting, for $1 \leq k \leq n$, the Gauss equation yields

$$\langle R(X_k, Y)Z, X_k \rangle = \langle \overline{R}(X_k, Y)Z, X_k \rangle + \lambda_k \langle AY, Z \rangle - \lambda_k^2 \langle X_k, Z \rangle \langle X_k, Y \rangle. \tag{2}$$

Thus, for the Ricci tensor Ric of Σ one has

$$\text{Ric}(Y, Z) = \sum_{k=1}^{n} \langle \overline{R}(X_k, Y)Z, X_k \rangle + H \langle AY, Z \rangle - \sum_{k=1}^{n} \lambda_k^2 \langle X_k, Z \rangle \langle X_k, Y \rangle, \quad (3)$$

where $H = \text{trace } A$ is the (non normalized) mean curvature of Σ. In particular, if we set $Y = X_i$ and $Z = X_j$, we have

$$\text{Ric}(X_i, X_j) = \sum_{k=1}^{n} \langle \overline{R}(X_k, X_i)X_j, X_k \rangle + H \delta_{ij} \lambda_i - \delta_{ij} \lambda_i^2. \quad (4)$$

2.3. Hypersurfaces in $I \times_\omega \mathbb{Q}_\epsilon^n$

In the above setting, let us consider the particular case when $\overline{M}^{n+1}$ is the warped product $I \times_\omega \mathbb{Q}_\epsilon^n$, where $I \subset \mathbb{R}$ is an open interval, ω is a positive differentiable function defined on I, and $\mathbb{Q}_\epsilon^n$ is one of the simply connected space forms of constant curvature $\epsilon \in \{0, 1, -1\}$: $\mathbb{R}^n$ ($\epsilon = 0$), $\mathbb{S}^n$ ($\epsilon = 1$), or $\mathbb{H}^n$ ($\epsilon = -1$).

Recall that the Riemannian metric of $I \times_\omega \mathbb{Q}_\epsilon^n$, to be denoted by $\langle\,,\,\rangle$, is:

$$dt^2 + \omega^2 ds_\epsilon^2,$$

where dt^2 and ds_ϵ^2 are the standard Riemannian metrics of $\mathbb{R}$ and $\mathbb{Q}_\epsilon^n$, respectively.

We call *horizontal* the fields of $T(I \times_\omega \mathbb{Q}_\epsilon^n)$ which are tangent to I, and *vertical* those which are tangent to $\mathbb{Q}_\epsilon^n$. The gradient of the projection π_I on the first factor I will be denoted by ∂_t.

Setting $\overline{\nabla}$ and $\widetilde{\nabla}$ for the Levi-Civita connections of $I \times_\omega \mathbb{Q}_\epsilon^n$ and $\mathbb{Q}_\epsilon^n$, respectively, for any *vertical* fields $X, Y \in T\mathbb{Q}_\epsilon^n$, the following identities hold (see [1, Lema 7.3]):

$$\begin{aligned}
\overline{\nabla}_X Y &= \widetilde{\nabla}_X Y - (\omega'/\omega)\langle X, Y \rangle \partial_t. \\
\overline{\nabla}_X \partial_t &= \overline{\nabla}_{\partial_t} X = (\omega'/\omega) X. \\
\overline{\nabla}_{\partial_t} \partial_t &= 0.
\end{aligned} \quad (5)$$

The curvature tensor $\overline{R}$ of $I \times_\omega \mathbb{Q}_\epsilon^n$ is given by (cf. [10])

$$\begin{aligned}
\langle \overline{R}(X, Y)Z, W \rangle = {}& (\alpha \circ \pi_I)(\langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle) \\
&+ (\beta \circ \pi_I)(\langle X, Z \rangle \langle Y, \partial_t \rangle \langle W, \partial_t \rangle - \langle Y, Z \rangle \langle X, \partial_t \rangle \langle W, \partial_t \rangle \\
&- \langle X, W \rangle \langle Y, \partial_t \rangle \langle Z, \partial_t \rangle + \langle Y, W \rangle \langle X, \partial_t \rangle \langle Z, \partial_t \rangle), \quad (6)
\end{aligned}$$

where $X, Y, Z, W \in T(I \times_\omega \mathbb{Q}_\epsilon^n)$, and α and β are the following functions on I:

$$\alpha := \frac{(\omega')^2 - \epsilon}{\omega^2} \quad \text{and} \quad \beta := \frac{\omega''}{\omega} - \alpha. \quad (7)$$

It is easily seen from (6) that $I \times_\omega \mathbb{Q}_\epsilon^n$ has constant sectional curvature if and only if α is constant and β vanishes on I. Also, a direct computation yields:

$$\alpha' = \frac{2\omega'}{\omega}\beta. \tag{8}$$

Given a hypersurface Σ of $I \times_\omega \mathbb{Q}_\epsilon^n$, the *height* function ξ and the *angle* function θ of Σ are defined as

$$\xi := \pi_I|_\Sigma \quad \text{and} \quad \Theta(x) := \langle N(x), \partial_t \rangle, \; x \in \Sigma.$$

The gradient of ξ on Σ is denoted by T, so that

$$T = \partial_t - \Theta N. \tag{9}$$

Definition 1. A connected Einstein hypersurface Σ of $I \times_\omega \mathbb{Q}_\epsilon^n$ on which $(\beta \circ \xi)T$ never vanishes will be called *ideal*.

Regarding the gradient of θ, let us notice first that, from the equalities (5), for all $X \in T(I \times_\omega \mathbb{Q}_\epsilon^n)$, one has

$$\overline{\nabla}_X \partial_t = \overline{\nabla}_{X - \langle X, \partial_t \rangle \partial_t} \partial_t = \frac{\omega'}{\omega}\left(X - \langle X, \partial_t \rangle \partial_t\right), \tag{10}$$

where, by abuse of notation, we are writing $\omega \circ \xi = \omega$ and $\omega' \circ \xi = \omega'$. Therefore, for any $X \in T\Sigma$, we have

$$X(\theta) = \langle \overline{\nabla}_X N, \partial_t \rangle + \langle N, \overline{\nabla}_X \partial_t \rangle = -\langle AX, T \rangle - \frac{\omega'}{\omega}\theta\langle T, X \rangle$$

$$= -\langle AT, X \rangle - \frac{\omega'}{\omega}\theta\langle T, X \rangle.$$

Hence, the gradient of θ is

$$\nabla\theta = -\left(A + \frac{\omega'}{\omega}\theta\,\mathrm{Id}\right)T, \tag{11}$$

where Id stands for the identity map of $T\Sigma$.

The following concept will play a fundamental role in this paper.

Definition 2. We say that a hypersurface Σ of $I \times_\omega \mathbb{Q}_\epsilon^n$ has the *T-property* if T is a principal direction of Σ, i.e., T never vanishes and there is a differentiable real function λ on Σ such that $AT = \lambda T$, where A is the shape operator of Σ.

Finally, it follows from (6) that, for any hypersurface Σ of $I \times_\omega \mathbb{Q}_\epsilon^n$, the Gauss equation (1) takes the form:

$$\begin{aligned}
\langle R(X,Y)Z, W \rangle = &(\alpha \circ \xi)(\langle X, Z \rangle\langle Y, W \rangle - \langle X, W \rangle\langle Y, Z \rangle)\\
&+(\beta \circ \xi)(\langle X, Z \rangle\langle Y, T \rangle\langle W, T \rangle - \langle Y, Z \rangle\langle X, T \rangle\langle W, T \rangle\\
&-\langle X, W \rangle\langle Y, T \rangle\langle Z, T \rangle + \langle Y, W \rangle\langle X, T \rangle\langle Z, T \rangle)\\
&+\langle AX, W \rangle\langle AY, Z \rangle - \langle AX, Z \rangle\langle AY, W \rangle.
\end{aligned} \tag{12}$$

3. Three Key Lemmas

Given a hypersurface $\Sigma \subset I \times_\omega \mathbb{Q}_\epsilon^n$, any nonempty transversal intersection

$$\Sigma_t := \Sigma \pitchfork (\{t\} \times \mathbb{Q}_\epsilon^n)$$

is a hypersurface of $(\{t\} \times \mathbb{Q}_\epsilon^n, ds_\epsilon^2)$ which we call the *vertical section* of Σ at height t. If Σ is oriented with unit normal N, it is easily seen that $N - \theta \partial_t$ is tangent to $\{t\} \times \mathbb{Q}_\epsilon^n$ and orthogonal to Σ_t, so that Σ_t is orientable as well. Denoting also by $\langle \, , \, \rangle_\epsilon$ the Riemannian metric ds_ϵ^2 of $\mathbb{Q}_\epsilon^n$, one has

$$\omega^2 \langle N - \theta \partial_t, N - \theta \partial_t \rangle_\epsilon = \langle N - \theta \partial_t, N - \theta \partial_t \rangle = 1 - \theta^2 = \|T\|^2.$$

Hence, a unit normal to Σ_t in $(\{t\} \times \mathbb{Q}_\epsilon^n, ds_\epsilon^2)$ is given by

$$N_t = -\frac{\omega(t)(N - \theta \partial_t)}{\|T\|}. \tag{13}$$

We shall denote by A_t the shape operator of Σ_t with respect to N_t, that is,

$$A_t X = -\widetilde{\nabla}_X N_t, \quad X \in T\Sigma_t \subset T\Sigma,$$

where $\widetilde{\nabla}$ stands for the Levi-Civita connection of $(\mathbb{Q}_\epsilon^n, ds_\epsilon^2)$.

Our first lemma establishes the relation between the shape operator A of a hypersurface $\Sigma \subset I \times_\omega \mathbb{Q}_\epsilon^n$ with the shape operator A_t of a vertical section $\Sigma_t \subset (\{t\} \times \mathbb{Q}_\epsilon^n, ds_\epsilon^2)$.

Lemma 1. *For $n \geq 2$, let Σ be a hypersurface of $I \times_\omega \mathbb{Q}_\epsilon^n$, and let Σ_t be one of its vertical sections (considered as a hypersurface of $(\{t\} \times \mathbb{Q}_\epsilon^n, ds_\epsilon^2)$). Then, the following identity holds:*

$$\langle A_t X, Y \rangle = -\frac{\omega}{\|T\|} \left(\langle AX, Y \rangle + \frac{\theta \omega'}{\omega} \langle X, Y \rangle \right) \quad \forall X, Y \in T\Sigma_t \subset T\Sigma. \tag{14}$$

Consequently, if T is a principal direction of Σ along Σ_t with corresponding principal curvature λ_1, the principal curvatures $\lambda_2, \ldots, \lambda_n$ of Σ are given by

$$\lambda_i = -\frac{\|T\|\lambda_i^t + \omega' \theta}{\omega}, \quad i = 2, \ldots, n, \tag{15}$$

where λ_i^t denotes the i-th principal curvature of Σ_t.

Proof. Given $X \in T\Sigma_t$, it follows from the first identity (5) that $\widetilde{\nabla}_X N_t = \overline{\nabla}_X N_t$, for $\langle X, N_t \rangle = 0$. However, from (13), we have

$$-\overline{\nabla}_X N_t = X(\omega/\|T\|)(N - \theta \partial_t) + (\omega/\|T\|)(\overline{\nabla}_X N - X(\theta)\partial_t - \theta \overline{\nabla}_X \partial_t).$$

Thus, since $\overline{\nabla}_X \partial_t = (\omega'/\omega)X$ and $AX = -\overline{\nabla}_X N$, for all $Y \in T\Sigma_t$, one has

$$\langle A_t X, Y \rangle = -\langle \widetilde{\nabla}_X N_t, Y \rangle = -\langle \overline{\nabla}_X N_t, Y \rangle = -\frac{\omega}{\|T\|} \langle AX + \theta(\omega/\omega')X, Y \rangle,$$

as we wished to prove.

Now, assuming that T is a principal direction of Σ along Σ_t, we have that $T\Sigma_t$ is invariant by A. This, together with (14), then yields

$$A_t = -\frac{\omega}{\|T\|}\left(A + \theta(\omega'/\omega)\mathrm{Id}\right)|_{T\Sigma_t},$$

where Id denotes the identity map of $T\Sigma_t$.

From this last equality, one easily concludes that any principal direction X_i of Σ_t with corresponding principal curvature λ_i^t is a principal direction of Σ with corresponding principal curvature λ_i as in (15). $\square$

Now, we establish a necessary and sufficient condition for an arbitrary hypersurface of $I \times_\omega \mathbb{Q}_\epsilon^n$ to have the T-property.

Lemma 2. *For $n > 2$, let Σ be a hypersurface of $I\times_\omega \mathbb{Q}_\epsilon^n$ on which $(\beta\circ\xi)T$ never vanishes. Then, Σ has the T-property if and only if the principal directions $X_1,\dots,X_n$ of Σ diagonalize its Ricci tensor. In particular, if Σ is Einstein, it has the T-property.*

Proof. Choosing $i,j,k \in \{1,\dots,n\}$ with $i \neq j \neq k \neq i$, we have from (6) that

$$\langle \overline{R}(X_k, X_i)X_j, X_k\rangle = -\beta\langle X_i, T\rangle\langle X_j, T\rangle.$$

Combining this equality with (4), we get

$$\mathrm{Ric}(X_i, X_j) = \sum_{k=1}^n \langle \overline{R}(X_k, X_i)X_j, X_k\rangle = (2-n)(\beta \circ \xi)\langle X_i, T\rangle\langle X_j, T\rangle,$$

from which the result follows. $\square$

Finally, as a direct consequence of equation (4), we obtain necessary and sufficient conditions on the principal curvatures of a hypersurface $\Sigma \subset I \times_\omega \mathbb{Q}_\epsilon^n$ with the T-property to be an Einstein manifold.

Lemma 3. *For $n > 2$, let Σ be a hypersurface of $I\times_\omega \mathbb{Q}_\epsilon^n$ having the T-property. Let $\lambda_1,\dots,\lambda_n$ be the principal curvatures of Σ, and let $\{X_1,\dots,X_n\}$ be the corresponding orthonormal frame of principal directions with $X_1 = T/\|T\|$. Then, Σ is a Λ-Einstein hypersurface if and only if the following equalities hold:*

$$\lambda_1^2 - H\lambda_1 + (n-1)((\beta\circ\xi)\|T\|^2 + (\alpha\circ\xi)) + \Lambda = 0,$$
$$\lambda_i^2 - H\lambda_i + (\beta\circ\xi)\|T\|^2 + (n-1)(\alpha\circ\xi) + \Lambda = 0, \quad 2 \leq i \leq n. \quad (16)$$

Proof. The proof is straightforward considering the identity (4) for $X_i = X_j = X_1$ and $X_i = X_j \neq X_1$, respectively. $\square$

TABLE 1. Definition of f

$f(x)$	ϵ	Rotational type
$\cos x$	1	Spherical
$\sinh x$	-1	Spherical
x	0	Spherical
x	-1	Parabolic
$\cosh x$	-1	Hyperbolic

4. Rotational CSC Hypersurfaces of $I \times_\omega \mathbb{Q}_\epsilon^n$

In [5], the authors introduced a class of hypersurfaces, called *rotational*, which was considered in [12] to construct and classify CSC hypersurfaces of $\mathbb{R} \times \mathbb{Q}_\epsilon^n$ (interchanging factors). Such a rotational hypersurface Σ is characterized by the fact that the horizontal projections on $\mathbb{Q}_\epsilon^n$ of its vertical sections constitute a parallel family $\mathscr{F}$ of totally umbilical hypersurfaces. In this way, Σ is called *spherical* if the members of $\mathscr{F}$ are geodesic spheres of $\mathbb{Q}_\epsilon^n$, *parabolic* if they are horospheres of $\mathbb{H}^n$, and *hyperbolic* if they are equidistant hypersurfaces of $\mathbb{H}^n$.

Given an interval $I \subset \mathbb{R}$, any rotational hypersurface $\Sigma \subset I \times \mathbb{Q}_\epsilon^n$ can be parametrized by means of a plane curve defined by two differentiable functions on an open interval. Here, we denote these functions by $\phi = \phi(s)$ and $\xi = \xi(s)$ (see [12] for details). In this setting, we can replace $I \times \mathbb{Q}_\epsilon^n$ by $I \times_\omega \mathbb{Q}_\epsilon^n$ and consider Σ as a hypersurface of the latter. Then, assuming that either

$$(\xi')^2 + (\omega(\xi)\phi')^2 = 1 \quad \text{or} \quad (\xi')^2 + \left(\omega(\xi)\frac{\phi'}{\phi}\right)^2 = 1 \text{ (if } \Sigma \text{ is parabolic),}$$

and performing computations analogous to the ones in [12, Section 4], one concludes that the metric $d\sigma^2$ on Σ induced by the warped metric of $I \times_\omega \mathbb{Q}_\epsilon^n$ is a warped metric as well, which is given by

$$d\sigma^2 = ds^2 + (\omega(\xi)f(\phi))^2 du^2, \tag{17}$$

where f, as defined in Table 1, depends on ϵ and the rotational type of Σ, and du^2 is the standard metric of $\mathbb{S}^{n-1}$, $\mathbb{R}^{n-1}$ or $\mathbb{H}^{n-1}$, according to whether Σ is spherical, parabolic or hyperbolic, respectively.

Let Σ be a rotational hypersurface of $I \times_\omega \mathbb{Q}_\epsilon^n$ whose metric $d\sigma^2$ is given by (17). By [12, Proposition 4.6], Σ has constant sectional curvature $c \in \mathbb{R}$ if its warping function $(\omega \circ \xi)f(\phi)$ coincides with the function $\psi = \psi(s)$, defined in Table 2, which depends on ϵ, the sign of c, and the rotational type of Σ.

Therefore, given $c \in \mathbb{R}$, our problem of finding a rotational hypersurface of $I \times_\omega \mathbb{Q}_\epsilon^n$ with constant sectional curvature c reduces to solving either the systems

TABLE 2. Definition of ψ

$\psi(s)$	sign of c	ϵ	Rotational type
$\frac{1}{\sqrt{c}} \sin \sqrt{c}s$	$c > 0$	$0, \pm 1$	Spherical
s	$c = 0$	$0, \pm 1$	Spherical
$\frac{1}{\sqrt{-c}} \sinh \sqrt{-c}s$	$c < 0$	$0, -1$	Spherical
$e^{\sqrt{-c}s}$	$c < 0$	-1	Parabolic
cte	$c = 0$	-1	Parabolic
$\frac{1}{\sqrt{-c}} \cosh \sqrt{-c}s$	$c < 0$	-1	Hyperbolic

$$\begin{cases} (\xi')^2 + (\omega(\xi)\phi')^2 = 1 \\ \omega(\xi)f(\phi) = \psi \end{cases} \quad \text{or} \quad \begin{cases} (\xi')^2 + \left(\omega(\xi)\frac{\phi'}{\phi}\right)^2 = 1 \\ \omega(\xi)f(\phi) = \psi, \end{cases} \tag{18}$$

where the second one is considered only in the parabolic case.

Let us assume that $\omega : I \to \omega(I)$ is a diffeomorphism, and that $f(\phi)$ does not vanish. Under these assumptions, the second equation of these systems gives that

$$\xi = \chi \left(\frac{\psi}{f(\phi)} \right), \tag{19}$$

where $\chi : \omega(I) \to I$ is the inverse of ω. Of course, equality (19) makes sense only if the range of $\psi/f(\phi)$ is contained in $\omega(I)$. Assuming that, taking the derivative, and substituting ξ' in the first equation of the system, we have that ϕ must be a solution of a nonlinear first order ODE in the following form:

$$a_2(s, y)(y')^2 + a_1(s, y)y' + a_0(s, y) = 0, \tag{20}$$

where each coefficient a_i is a differentiable function.

Let us assume that, for some pair (s_0, y_0), we have

$$\Delta(s_0, y_0) := a_1^2(s_0, y_0) - 4a_2(s_0, y_0)a_0(s_0, y_0) > 0. \tag{21}$$

In this case, (20) can be written as

$$y' = F(s, y), \tag{22}$$

where F is a differentiable function in a neighborhood of (s_0, y_0).

Therefore, if the inequality (21) holds, we can apply Picard's Theorem and conclude that there exists a solution $y = \phi(s)$ to (22), and so to (20), satisfying the initial condition $\phi(s_0) = y_0$ (see [8, pg 82] for details). Applying it to (19), we obtain the function ξ, so that (ϕ, ξ) is a solution of (18).

The following result, which will be considered in the proof of Theorem 1, summarizes the above discussion.

Proposition 1. *Let $\omega : I \to \omega(I)$ be a diffeomorphism. Given $c \in \mathbb{R}$, suppose that there exists $s_0 \in \mathbb{R}$ such that the function ξ given in (24) is well defined in an open interval $I_0 \ni s_0$. If, for some $y_0 \in \mathbb{R}$, one has $\Delta(s_0, y_0) > 0$, then the system (18) has a solution (ϕ, ξ) defined in an open interval contained in I_0. Consequently, there is a hypersurface of constant sectional curvature c in the corresponding warped product $I \times_\omega \mathbb{Q}_\epsilon^n$.*

Proof of Theorem 1. We can assume, without loss of generality, that

$$\omega(I) = (0, \delta), \ \ 0 < \delta \leq +\infty,$$

for $\omega > 0$. Then, the hypothesis on ω' yields

$$|\chi'(u)| < 1 \ \ \forall u \in (0, \delta). \tag{23}$$

We will consider separately the cases in the statement (according to ϵ, c, and the rotational type of the hypersurface) to show that, in any of them, there exists a pair (s_0, y_0) such that $\xi(s)$ (as given in (19)) is well defined in an open interval $I_0 \ni s_0$, and $\Delta(s_0, y_0) > 0$. The result, then, will follow from Proposition 1.

    Case 1: $\underline{\epsilon = -1, c \in (-\infty, +\infty), \text{ spherical-type}}$

In this setting, $f(x) = \sinh x$ and $\psi(s)$ is as in the first three lines of Table 2, according to the sign of c. Then, we have

$$\xi(s) = \begin{cases} \chi\left(\frac{\sin(\sqrt{c}s)}{\sqrt{c}\sinh(\phi(s))}\right) & \text{if } c > 0. \\ \chi\left(\frac{s}{\sinh(\phi(s))}\right) & \text{if } c = 0. \\ \chi\left(\frac{\sinh(\sqrt{-c}s)}{\sqrt{-c}\sinh(\phi(s))}\right) & \text{if } c < 0. \end{cases} \tag{24}$$

In any case, we choose s_0 positive and sufficiently small, and $y_0 = \phi(s_0)$ satisfying $\sinh y_0 > 1$, so that ξ is well defined in an open interval $I_0 \ni s_0$, $I_0 \subset (0, +\infty)$.

A straightforward calculation gives that, at (s_0, y_0), the coefficients a_i (defined in (20)) satisfy $a_2(s_0, y_0) > 0$ and

$$a_0(s_0, y_0) = \begin{cases} (\chi'(u_0))^2 \frac{\cos^2(\sqrt{c}s_0)}{\sinh^2(y_0)} - 1 & \text{if } c > 0. \\ \frac{(\chi'(u_0))^2}{\sinh^2(y_0)} - 1 & \text{if } c = 0. \\ (\chi'(u_0))^2 \frac{\cosh^2(\sqrt{-c}s_0)}{\sinh^2(y_0)} - 1 & \text{if } c < 0, \end{cases}$$

where u_0 is the argument of χ in (24) for $s = s_0$. From the choice of y_0 and (23), we have $a_0(s_0, y_0) < 0$ in all three cases, which implies that $\Delta(s_0, y_0) > 0$.

    Case 2: $\underline{\epsilon = 0, c \in (-\infty, +\infty), \text{spherical-type}}$

The reasoning for this case is completely analogous to the one for Case 1. We have just to replace, in that argument, the function $f(x) = \sinh x$ by the identity function $f(x) = x$.

    Case 3: $\underline{\epsilon = 1, c > 0, \text{spherical-type}}$

Now, the functions f and ψ are $f(x) = \cos x$ and $\psi(s) = \sin(\sqrt{c}s)/\sqrt{c}$, in which case equality (19) takes the form

$$\xi(s) = \chi\left(\frac{\sin(\sqrt{c}s)}{\sqrt{c}\cos\phi(s)}\right).$$

Hence, choosing s_0 and $y_0 = \phi(s_0)$ positive and sufficiently small, one has

$$u_0 := \frac{\sin(\sqrt{c}s_0)}{\sqrt{c}\cos y_0} \in \omega(I),$$

which implies that $\xi(s)$ is well defined in an open interval $I_0 \ni s_0$, $I_0 \subset (0, +\infty)$. Also, a direct computation yields $a_2(s_0, y_0) > 0$ and

$$a_0(s_0, y_0) = (\chi'(u_0))^2 \frac{\cos^2(\sqrt{c}s_0)}{\cos^2(y_0)} - 1.$$

Thus, assuming $y_0 < \sqrt{c}s_0$ and considering (23), we conclude that $a_0(s_0, y_0) < 0$. In particular, $\Delta(s_0, y_0) > 0$.

Case 4: $\epsilon = -1, c \leq 0$, parabolic-type

For this case, $f(x) = x$ and $\psi(s)$ is as in the fourth and fifth lines of Table 2, according to the sign of c. Hence,

$$\xi(s) = \begin{cases} \chi\left(\frac{e^{\sqrt{-c}s}}{\phi(s)}\right) & \text{if } c < 0. \\ \chi\left(\frac{a}{\phi(s)}\right) & \text{if } c = 0, \end{cases} \tag{25}$$

where a is a positive constant. Setting $s_0 = 0$, in any of the above cases, we can choose a sufficiently large $y_0 = \phi(0)$ such that ξ is well defined in a neighborhood of 0. For $c < 0$, we shall also assume $y_0 > -c$.

Writing u_0 for the argument of χ in (25), we have $a_2(s_0, y_0) > 0$ and

$$a_0(s_0, y_0) = \begin{cases} \frac{-c(\chi'(u_0))^2}{y_0^4} - 1 & \text{if } c < 0. \\ -1 & \text{if } c = 0. \end{cases}$$

So, considering (23) (only for $c < 0$), we have that $\Delta(s_0, y_0) > 0$ in both cases.

Case 5: $\epsilon = -1, c < 0$, hyperbolic-type The functions f and ψ are $f(x) = \cosh x$ and $\psi(s) = \cosh s$. Thus, we have

$$\xi(s) = \chi\left(\frac{\cosh(\sqrt{-c}s)}{\sqrt{-c}\cosh\phi(s)}\right). \tag{26}$$

As in the previous case, by choosing $s_0 = 0$ and $y_0 = \phi(s_0)$ sufficiently large, we have that ξ is well defined in a neighborhood of 0. Again, setting u_0 for the argument of ξ in (26), we have $a_2(s_0, y_0) > 0$ and

$$a_0(s_0, y_0) = \frac{(\chi'(u_0)\sinh(\sqrt{-c}s_0))^2}{\cosh^4(y_0)} - 1 = -1 < 0,$$

which implies that $\Delta(s_0, y_0) > 0$. □

Besides having totally umbilical vertical sections, a notable fact of rotational hypersurfaces in $I \times \mathbb{Q}_\epsilon^n$ is that they have the T-property (see [5]). Let us see that the same is true for hypersurfaces of warped products $I \times_\omega \mathbb{Q}_\epsilon^n$. To that end, we first consider the diffeomorphism $G : I \to J = G(I)$ such that $G' = 1/\omega$. As can be easily checked, the map

$$\begin{aligned} \varphi \colon I \times_\omega \mathbb{Q}_\epsilon^n &\to J \times \mathbb{Q}_\epsilon^n \\ (t, p) &\mapsto (G(t), p) \end{aligned} \tag{27}$$

is a conformal diffeomorphism. This, together with [13, Lemma 3], gives that a hypersurface $\Sigma \subset I \times_\omega \mathbb{Q}_\epsilon^n$ has the T-property if and only if $\varphi(\Sigma)$ has the T-property as a hypersurface of $J \times \mathbb{Q}_\epsilon^n$. Also, since φ fixes the second factor $\mathbb{Q}_\epsilon^n$ pointwise, a vertical section $\Sigma_t \subset \Sigma$ is totally umbilical in $(\{t\} \times \mathbb{Q}_\epsilon^n, ds_\epsilon^2)$ if and only if $\varphi(\Sigma_t) \subset \varphi(\Sigma)$ is totally umbilical in $((\{G(t)\} \times \mathbb{Q}_\epsilon^n, ds_\epsilon^2)$. Consequently, Σ is rotational if and only if $\varphi(\Sigma)$ is rotational.

The following result, which will be applied in the proof of Theorem 2, follows directly from the above considerations, the identity (15), and [5, Theorem 2] (see also [12, Remark 7.4]).

Proposition 2. *For $n \geq 3$, let $\Sigma \in I \times_\omega \mathbb{Q}_\epsilon^n$ be a hypersurface having the T-property. Then, Σ is rotational if and only if any vertical section Σ_t of Σ is totally umbilical in $(\{t\} \times \mathbb{Q}_\epsilon^n, ds_\epsilon^2)$.*

Proof of Theorem 2. As Σ has constant sectional curvature, it is an Einstein manifold. Thus, by Lemma 2, T is a principal direction of Σ, so that we can set

$$\{X_1, \ldots, X_n\} \subset T\Sigma$$

for the orthonormal frame of principal directions of Σ with $X_1 = T/\|T\|$.

Writing c for the sectional curvature of Σ, λ_1 for its principal curvature corresponding to T, and $\lambda_2, \ldots, \lambda_n$ for the principal curvatures corresponding to the vertical principal directions of Σ, we get from Gauss equation (12) that, for all $2 \leq i, j \leq n$, $i \neq j$, the following identities hold:

$$\begin{aligned} (\alpha \circ \xi) + c &= \lambda_i \lambda_j \\ (\alpha \circ \xi) + c &= \lambda_1 \lambda_i - (\beta \circ \xi)\|T\|^2. \end{aligned} \tag{28}$$

Suppose that, for some $x \in \Sigma$, $\alpha(\xi(x)) + c = 0$. Then, from the first identity in (28), $\lambda_i(x)$ is nonzero for at most one index $i \in \{2, \ldots, n\}$. Thus, choosing $i \in \{2, \ldots, n\}$ such that $\lambda_i(x) = 0$, we have from the second identity in (28) that $(\beta \circ \xi)\|T\|$ vanishes at x, which is contrary to our hypothesis. Hence, $(\alpha \circ \xi) + c$ never vanishes on Σ.

Since we are assuming $n > 3$, the first identity in (28) implies that the last $n - 1$ principal curvatures of Σ are all equal and nonzero. This, together with (15), gives that any vertical section Σ_t of Σ is totally umbilical in $(\{t\} \times \mathbb{Q}_\epsilon^n, ds_\epsilon^2)$. The result, then, follows from Proposition 2. $\qquad \square$

5. Proofs of Theorems 3 and 4

Proof of Theorem 3. The proof follows closely the one given for [14, Theorem3.1].

Considering equality (8) and the fact that $T = \nabla\xi$, it follows from the hypothesis on $(\beta \circ \xi)T$ that the derivative of $\alpha \circ \xi$ vanishes on Σ. Indeed, given $x \in \Sigma$, one has

$$(\alpha \circ \xi)_*(x) = \alpha'(\xi(x))\xi_*(x) = \frac{2\omega'(\xi(x))}{\omega(\xi(x))}\beta(\xi(x))\xi_*(x) = 0.$$

In particular, $\alpha \circ \xi$ (to be denoted by α) is constant, since Σ is connected.

Let $\{X_1, \ldots, X_n\} \subset T\Sigma$ be a local orthonormal frame of principal directions of Σ. It follows from (6) that $\langle \overline{R}(X_i, X_k)X_k, X_i \rangle = -\alpha \ \forall i \neq k \in \{1, \ldots, n\}$. This equality, together with (4), gives that any principal direction λ_i of Σ is a root of the following quadratic equation:

$$s^2 - Hs + \sigma = 0. \tag{29}$$

In particular, there are only two possible values for each λ_i. Hence, we can assume that, for some $k \in \{1, \ldots, n\}$, one has

$$\lambda_1 = \cdots = \lambda_k = \lambda \quad \text{and} \quad \lambda_{k+1} = \cdots = \lambda_n = \mu,$$

with λ and μ satisfying the identities:

$$\lambda + \mu = H \quad \text{and} \quad \lambda\mu = \sigma. \tag{30}$$

Since $H = \text{trace}\, A = k\lambda + (n-k)\mu$, it follows that

$$(k-1)\lambda + (n-k-1)\mu = 0. \tag{31}$$

Assume $\sigma > 0$. In this case, if λ and μ are distinct at some point of Σ, they must have the same sign. Thus, by equality (31), we must have $k-1 = n-k-1 = 0$, which gives $n = 2$. However, we are assuming $n > 2$. Hence, Σ is totally umbilical if $\sigma > 0$. In addition, $\lambda^2 = \sigma$, i.e., λ is constant on Σ. But, by Gauss equation (12), for any pair of orthonormal principal directions X_i, X_j, one has $K(X_i, X_j) = \lambda^2 - \alpha$, which implies that Σ has constant sectional curvature $c = \lambda^2 - \alpha$.

Suppose now that $\sigma = 0$. If $\lambda \neq 0$ and $\mu = 0$, we have $H = k\lambda$. However, λ is a solution of (29). Hence, $\lambda^2 - k\lambda^2 = 0$, that is, $k = 1$, which implies that Σ has at most one nonzero principal curvature. This, together with Gauss equation, gives that Σ has constant sectional curvature $c = -\alpha$.

Finally, assume that $\sigma = \lambda\mu < 0$, in which case λ and μ are distinct and nonzero. Then, considering (31), we conclude that $1 < k < n-1$, that is, both λ and μ have multiplicity at least 2. In addition, since H is differentiable on Σ, as distinct roots of (29), λ and μ are also differentiable. This, together with (31), gives that k is differentiable, and so is constant on Σ, that is, λ and μ have constant multiplicities.

It follows from the above considerations and the identities (30) and (31) that λ and μ are constant functions on Σ. Now, choosing orthonormal pairs

X_1, X_2 and Y_1, Y_2 satisfying $AX_i = \lambda X_i$ and $AY_i = \mu Y_i$, we get from Gauss equation that

$$K(X_1, X_2) = \lambda^2 - \alpha \neq \mu^2 - \alpha = K(Y_1, Y_2),$$

which implies that Σ has nonconstant sectional curvature. $\square$

Remark 1. It is a well known fact that, for any $\delta \in \{-1, 0, 1\}$, $\mathbb{Q}_\delta^{n+1}$ contains open dense subsets which can be represented as warped products $I \times_\omega \mathbb{Q}_\epsilon^n$ (see, e.g., [13, Section 3.2]). In all these representations, the corresponding function α is constant, so that Theorem 3 applies. More precisely, the cases (i) and (ii) occur for any $\delta \in \{-1, 0, 1\}$, whereas case (iii) occurs only for $\delta = 1$. Namely, for the set $\mathbb{S}^{n+1} - \{e_{n+2}, -e_{n+2}\} = (0, \pi) \times_{\sin t} \mathbb{S}^n$ (cf. [14, Theorem 3.1]).

Let us recall that a Riemannian manifold $\overline{M}^{n+1}$ is called *conformally flat* if it is locally conformal to Euclidean space. Riemannian manifolds of constant sectional curvature are known to be conformally flat. Conversely, any conformally flat Einstein manifold has constant sectional curvature. Another classical result on this subject states that, for $n \geq 4$, any hypersurface of a conformally flat manifold $\overline{M}^{n+1}$ is also conformally flat if and only if it has a principal curvature of multiplicity $n-1$ (cf. [4, chap. 16]). These facts will be considered in the proof below.

Proof of Theorem 4. By Lemma 2, Σ has the T-property. Hence, by Proposition 2, the assertions (ii) and (iii) are equivalent. In addition, by Theorem 2, (i) implies (iii). So, it remains to prove that (ii) implies (i). To that end, let us observe that, from the second equality in (16), any of the principal curvatures λ_i, $i \in \{2, \ldots, n\}$, of Σ is a root of the quadratic equation $s^2 - Hs + \sigma = 0$, where

$$\sigma = \Lambda + (n-1)(\alpha \circ \xi) + \|T\|^2 (\beta \circ \xi). \tag{32}$$

Therefore, at each point of Σ, there are at most two different values for such principal curvatures, that is, there exists $k \in \{2, \ldots, n\}$ such that

$$\lambda_2 = \cdots = \lambda_k = \lambda \quad \text{and} \quad \lambda_{k+1} = \cdots = \lambda_n = \mu.$$

Consider a vertical section $\Sigma_t \subset \Sigma$, $t \in I$. It follows from Lemma 1 that $X_2, \ldots, X_n$ are principal directions of Σ_t whose principal curvatures are

$$\lambda_t := -\frac{\omega(t)}{\|T\|}\left(\lambda + \frac{\theta\omega'(t)}{\omega(t)}\right) \quad \text{and} \quad \mu_t := -\frac{\omega(t)}{\|T\|}\left(\mu + \frac{\theta\omega'(t)}{\omega(t)}\right). \tag{33}$$

In particular, $\lambda_t = \mu_t$ if and only if $\lambda = \mu$.

Thus, if all vertical sections $\Sigma_t \subset \Sigma$ are totally umbilical, then Σ has a principal curvature of multiplicity $n-1$. Hence, Σ is conformally flat, since $I \times_\omega \mathbb{Q}_\epsilon^n$ is conformally flat (cf. [4, Examples 16.2]). Being Einstein and conformally flat, Σ has constant sectional curvature, and so is trivial. This shows that (ii) implies (i) and finishes the proof of the theorem. $\square$

6. Einstein Graphs in $I \times_\omega \mathbb{Q}_\epsilon^n$

Consider an oriented isometric immersion

$$f : M_0^{n-1} \to \mathbb{Q}_\epsilon^n,$$

of a Riemannian manifold M_0 into $\mathbb{Q}_\epsilon^n$. Suppose that there exists a neighborhood $\mathscr{U}$ of M_0 in $T^\perp M_0$ without focal points of f, that is, the restriction of the normal exponential map $\exp_{M_0}^\perp : T^\perp M_0 \to M$ to $\mathscr{U}$ is a diffeomorphism onto its image. Denoting by η the unit normal field of f, there is an open interval $I \ni 0$ such that, for all $p \in M_0$, the curve

$$\gamma_p(s) = \exp_{\mathbb{Q}_\epsilon^n}(f(p), s\eta(p)), \ s \in I, \tag{34}$$

is a well defined geodesic of $\mathbb{Q}_\epsilon^n$ without conjugate points. Thus, for all $s \in I$,

$$f_s : M_0 \to \mathbb{Q}_\epsilon^n$$
$$p \mapsto \gamma_p(s)$$

is an immersion of M_0 into $\mathbb{Q}_\epsilon^n$, which is said to be *parallel* to f. Notice that, given $p \in M_0$, the tangent space $f_{s_*}(T_p M_0)$ of f_s at p is the parallel transport of $f_*(T_p M_0)$ along γ_p from 0 to s. We also remark that, with the induced metric, the unit normal η_s of f_s at p is given by

$$\eta_s(p) = \gamma_p'(s).$$

Definition 3. Let $\phi : I \to \phi(I) \subset \mathbb{R}$ be an increasing diffeomorphism, i.e., $\phi' > 0$. With the above notation, we call the set

$$\Sigma := \{(\phi(s), f_s(p)) \in I \times_\omega \mathbb{Q}_\epsilon^n \, ; \, p \in M_0, \, s \in I\}, \tag{35}$$

the *graph* determined by ϕ and $\{f_s \, ; \, s \in I\}$ or (ϕ, f_s)-*graph*, for short.

Let $\Sigma \subset I \times_\omega \mathbb{Q}_\epsilon^n$ be a (ϕ, f_s)-graph with the induced metric from $I \times_\omega \mathbb{Q}_\epsilon^n$. For an arbitrary point $x = (\phi(s), f_s(p))$ of Σ, one has

$$T_x \Sigma = f_{s_*}(T_p M_0) \oplus \mathrm{Span}\{\partial_s\}, \ \partial_s = \eta_s + \phi'(s)\partial_t,$$

where η_s denotes the unit normal field of f_s.

Writing, by abuse of notation, $\omega \circ \phi = \omega$, a unit normal to Σ is

$$N = \frac{-\phi'/\omega}{\sqrt{1 + (\phi'/\omega)^2}} \frac{\eta_s}{\omega} + \frac{1}{\sqrt{1 + (\phi'/\omega)^2}} \partial_t. \tag{36}$$

In particular, the angle function of Σ is

$$\Theta = \frac{1}{\sqrt{1 + (\phi'/\omega)^2}} \cdot \tag{37}$$

So, if we set

$$\rho := \frac{\phi'/\omega}{\sqrt{1 + (\phi'/\omega)^2}}, \tag{38}$$

we can write N as

$$N = -\rho \frac{\eta_s}{\omega} + \theta \partial_t, \tag{39}$$

which yields $1 = \rho^2 + \theta^2$, i.e., $\rho = \|T\|$. It also follows from (39) that

$$\overline{\nabla}_{\partial_s} N = -\left(\frac{\rho}{\omega}\right)' \eta_s - \frac{\rho}{\omega}\overline{\nabla}_{\partial_s}\eta_s + \langle \nabla\theta, \partial_s\rangle \partial_t + \theta\overline{\nabla}_{\partial_s}\partial_t \,.$$

However, by the identities (5), setting $\zeta = \omega'/\omega$ and, again by abuse of notation, $\zeta \circ \phi = \zeta$, we have that

$$\begin{aligned}
\overline{\nabla}_{\partial_s}\eta_s &= \overline{\nabla}_{\eta_s}\eta_s + \phi'\overline{\nabla}_{\partial_t}\eta_s \\
&= \widetilde{\nabla}_{\eta_s}\eta_s - \zeta\langle\eta_s, \eta_s\rangle\partial_t + \phi'\zeta\eta_s \\
&= -\zeta\omega^2\partial_t + \phi'\zeta\eta_s.
\end{aligned}$$

and $\overline{\nabla}_{\partial_s}\partial_t = \overline{\nabla}_{\eta_s}\partial_t = \zeta\eta_s$. Therefore,

$$A\partial_s = -\overline{\nabla}_{\partial_s} N = -(\rho\zeta\omega + \langle\nabla\theta, \partial_s\rangle)\partial_t + \left(\frac{\rho'}{\omega} - \theta\zeta\right)\eta_s \,,$$

which implies that $A\partial_s$ is orthogonal to any vertical field $X \in \{\partial_s\}^\perp \subset T\Sigma$. Thus, ∂_s is a principal direction of Σ with eigenvalue

$$\lambda_1 = \frac{\rho'}{\omega} - \theta\zeta.$$

In particular, λ_1 is a function of s alone.

From (37), we have that $0 < \theta < 1$. Thus, T is non vanishing on Σ and parallel to ∂_s, which implies that T is also a principal direction of Σ with eigenvalue λ_1.

Now, let $X_2, \ldots, X_n$ be orthonormal vertical principal directions of Σ. By Lemma 1, these fields are also principal directions of the vertical sections $\Sigma_s = \Sigma_{\phi(s)}$ with corresponding principal curvatures

$$\lambda_i^s = -\frac{\omega}{\|T\|}(\lambda_i + \theta\zeta) = -\frac{\omega}{\rho}(\lambda_i + \theta\zeta), \quad i = 2, \ldots, n. \tag{40}$$

Summarizing, we have the following result.

Proposition 3. *Let Σ be a (ϕ, f_s)-graph in $I \times_\omega \mathbb{Q}_\epsilon^n$ with unit normal N (with respect to the induced metric) as in (36). Then, Σ has the T-property, and its principal curvatures are given by*

$$\lambda_1 = \frac{\rho'(s)}{\omega(\phi(s))} - \theta\zeta(\phi(s)), \quad \lambda_i = -\frac{\rho(s)}{\omega(\phi(s))}\lambda_i^s(p) - \theta\zeta(\phi(s)), \quad i = 2, \ldots, n, \tag{41}$$

where θ is as in (37), $\zeta = (\omega' \circ \phi)/(\omega \circ \phi)$, and $\lambda_i^s(p)$ is the i-th principal curvature of the parallel $f_s : M_0 \to \mathbb{Q}_\epsilon^n$ at $p \in M_0$.

It is well-known that the principal curvatures $\lambda_i^s(p)$ in the statement of Proposition 3 are given by (see [3,7]):

$$\lambda_i^s(p) = \frac{\epsilon S_\epsilon(s) + C_\epsilon(s)\lambda_i^0(p)}{C_\epsilon(s) - S_\epsilon(s)\lambda_i^0(p)}, \tag{42}$$

where λ_i^0 is the i-th principal curvature of $f = f_0$, and C_ϵ, S_ϵ are as in Table 3.

TABLE 3. Definition of C_ϵ and S_ϵ

Function	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = -1$
$C_\epsilon(s)$	1	$\cos s$	$\cosh s$
$S_\epsilon(s)$	s	$\sin s$	$\sinh s$

In particular, the following elementary equalities hold:

$$C'_\epsilon(s) = -\epsilon S_\epsilon(s)$$
$$S'_\epsilon(s) = C_\epsilon(s) \tag{43}$$
$$C^2_\epsilon(s) + \epsilon S^2_\epsilon(s) = 1.$$

From identities (41), (42) and (43), we have

$$\lambda_1 = -\frac{\theta}{\phi'}\frac{d}{ds}\log(\theta\omega), \tag{44}$$

$$\lambda_i^s(p) = -\frac{d}{ds}\log\left(C_\epsilon(s) - S_\epsilon(s)\lambda_i^0(p)\right), \tag{45}$$

$$\frac{d}{ds}\lambda_i^s(p) = \epsilon + (\lambda_i^s(p))^2. \tag{46}$$

Let $f : M_0 \to \mathbb{Q}_\epsilon^n$ be a hypersurface, and let

$$\mathscr{F} = \{f_s : M_0 \to \mathbb{Q}_\epsilon^n \,;\, s \in I \ni 0\}$$

be a family of parallel hypersurfaces to $f = f_0$. We say that f is *isoparametric* if any hypersurface f_s has constant mean curvature. It was proved by Cartan that f is isoparametric if and only if any f_s has constant principal curvatures λ_i^s (possibly depending on i and s).

Suppose that, for an integer $d > 1$, $\lambda_1, \ldots, \lambda_d$ are pairwise distinct principal curvatures of an isoparametric hypersurface $f : M_0 \to \mathbb{Q}_\epsilon^n$. The following formula, also due to Cartan, holds for any fixed $i \in \{1, \ldots, d\}$ (cf. [3,7]):

$$\sum_{i \neq j} n_j \frac{\epsilon + \lambda_i\lambda_j}{\lambda_i - \lambda_j} = 0,$$

where n_j denotes the multiplicity of λ_j.

If $n > 2$ and f has only two distinct principal curvatures, say λ and μ, we have from equality (42) that the same is true for each parallel f_s. Denoting their corresponding principal curvatures by λ^s and μ^s, Cartan's formula yields

$$\lambda^s\mu^s = -\epsilon \;\; \forall s \in I. \tag{47}$$

7. Proofs of Theorems 5–7

Proof of Theorem 5. By Lemma 2, Σ has the T-property. So, the same is true for $\varphi(\Sigma) \subset J \times \mathbb{Q}_\epsilon^n$, where φ is the map given in (27). Also, since φ is a conformal

diffeomorphism, the angle function of $\varphi(\Sigma)$ is non vanishing in $J \times \mathbb{Q}^n_\epsilon$. Hence, by [15, Theorem 1], $\varphi(\Sigma)$ is given locally by a $(\tilde{\phi}, f_s)$-graph, which implies that Σ is locally a (ϕ, f_s)-graph, where $\phi = G^{-1} \circ \tilde{\phi}$, and G is the diffeomorphism $G : I \to J$ such that $G' = 1/\omega$. This proves (i). (By abuse of notation, we shall call Σ such a local (ϕ, f_s)-graph.)

Let $\{X_1, \ldots, X_n\}$ be an orthonormal basis of principal directions of Σ such that $X_1 = T/\|T\|$. Since $\rho = \|T\|$, equations (16) become

$$
\begin{aligned}
\lambda_1^2 - H\lambda_1 + (n-1)(\beta\rho^2 + \alpha) + \Lambda &= 0 \\
\lambda_i^2 - H\lambda_i + \beta\rho^2 + (n-1)\alpha + \Lambda &= 0,
\end{aligned}
\tag{48}
$$

where, by abuse the notation, we write $\alpha \circ \xi = \alpha$ and $\beta \circ \xi = \beta$.

The second of the above equations gives that Σ has at most two distinct principal curvatures among the last $n-1$. Let us call them λ and μ. So, one has

$$
\begin{aligned}
\lambda + \mu &= H \\
\lambda\mu &= (n-1)(\alpha + \beta\rho^2) + \Lambda.
\end{aligned}
\tag{49}
$$

Let us assume first that λ_1 vanishes in an open interval $I_0 \subset I$. Denoting the multiplicity of λ and μ by n_λ and n_μ, respectively, it follows from the first equation in (48) that equations (49) can be rewritten as

$$
\begin{aligned}
(1-n_\lambda)\lambda + (1-n_\mu)\mu &= 0 \\
\lambda\mu &= -(n-2)\beta\rho^2.
\end{aligned}
\tag{50}
$$

Since $n > 2$ and $\beta\|T\|$ never vanishes, it follows from equations (50) that $\beta > 0$, and that λ and μ depend only on s. Since λ_1 also depends only on s, we conclude that f_s is isoparametric.

By continuity, we can assume now that λ_1 never vanishes. Since λ_1 and $\beta\rho^2 + \alpha$ depend only on s, it follows from the first equality in (48) that the same is true for H. This, together with (40), gives that the mean curvature H_s of f_s is a function of s alone, which implies that the family $\{f_s\}$ is isoparametric. In addition, identity (42) gives that f_s, just as $f = f_0$, has at most two distinct principal curvatures, which we shall denote by λ^s and μ^s. This finishes the proof of (ii).

To prove (iii), let us suppose by contradiction that $\lambda \neq \mu$ and, consequently, that $\lambda^s \neq \mu^s$. Since we are assuming $\lambda_1 \equiv 0$, it follows from (41) that

$$
\begin{aligned}
\theta\zeta &= \frac{\rho'}{\omega} \\
\lambda &= -\frac{\rho\lambda^s + \rho'}{\omega} \\
\mu &= -\frac{\rho\mu^s + \rho'}{\omega}.
\end{aligned}
\tag{51}
$$

In particular, Eq. (50) can be written as

$$(n_\lambda - 1)\lambda^s + (n_\mu - 1)\mu^s + (n - 3)\frac{\rho'}{\rho} = 0$$

$$\left(\frac{\rho'}{\rho}\right)^2 + (\lambda^s + \mu^s)\frac{\rho'}{\rho} + (n - 3)\epsilon + (n - 2)\omega^2\zeta' = 0. \tag{52}$$

Now, equation (44) yields $\theta\omega = \sqrt{c}$ for some positive constant c. Therefore, from the first equation in (51), we have $\zeta = \rho'/\sqrt{c}$. Moreover, from the expressions of Θ and ρ in (37) and (38), one has $\rho/\theta = \phi'/\omega$. Hence,

$$\omega^2\zeta' = \frac{\omega^2\rho''}{\phi'\sqrt{c}} = \frac{\omega\rho''}{\phi'\Theta} = \frac{\rho''}{\rho},$$

so that

$$\omega^2\zeta' = \left(\frac{\rho'}{\rho}\right)' + \left(\frac{\rho'}{\rho}\right)^2.$$

From this and the second identity in (52), we have

$$(n - 1)\left(\frac{\rho'}{\rho}\right)^2 + (\lambda^s + \mu^s)\frac{\rho'}{\rho} + (n - 3)\epsilon + (n - 2)\left(\frac{\rho'}{\rho}\right)' = 0. \tag{53}$$

Using (46) and the first equality in (52), we also have

$$\left(\frac{\rho'}{\rho}\right)' = -\epsilon + \left(\frac{1 - n_\lambda}{n - 3}\right)(\lambda^s)^2 + \left(\frac{1 - n_\mu}{n - 3}\right)(\mu^s)^2. \tag{54}$$

Finally, Cartan's formula $\lambda^s\mu^s = -\epsilon$ and equalities (52)–(54) yield

$$\frac{(n - 1)(1 - n_\lambda)(1 - n_\mu)}{(n - 3)^2}\left((\lambda^s)^2 + (\mu^s)^2 + 2\epsilon\right) = 0.$$

Since, from (50), we have $n_\lambda > 1$ and $n_\mu > 1$, the above equation reduces to $(\lambda^s)^2 + (\mu^s)^2 + 2\epsilon = 0$. Again from $\lambda^s\mu^s = -\epsilon$, we have $(\lambda^s - \mu^s)^2 = 0$, which yields $\lambda^s \equiv \mu^s$, so that $\lambda \equiv \mu$. Therefore, all vertical sections of Σ are totally umbilical. So, Theorem 4 applies and gives that Σ is trivial. This shows (iii) and finishes the proof of the theorem. $\qquad\square$

Proof of Theorem 6. Since $\theta < 1$ is constant, we have that T never vanishes on Σ and, from (11), that T is a principal direction with associated principal curvature

$$\lambda_1 = -\theta\frac{\omega'}{\omega}. \tag{55}$$

Let us suppose that $\beta \circ \xi$ vanishes in a connected open set $\Sigma' \subset \Sigma$. Then, from Theorem 3, $\alpha \circ \xi$ is constant on Σ', and Σ' has constant sectional curvature, unless $\sigma := \Lambda + (n - 1)(\alpha \circ \xi)$ is negative. In this case, Theorem 3 also gives that all principal curvatures of Σ' are constant. Hence, from (55), ω'/ω is constant on Σ'. Since $\alpha \circ \xi$ is also constant, we have from (7) that ω is

constant on Σ', which implies that λ_1 vanishes there. However, from Lemma 3, we have

$$0 = \lambda_1^2 - H\lambda_1 + \sigma = \sigma < 0,$$

which is clearly a contradiction.

From the above considerations and the connectedness of Σ, we can assume that $\beta \circ \xi$ never vanishes on Σ, which makes it an ideal Einstein hypersurface. Assuming also that $\theta > 0$, we have from Theorem 5 that Σ is locally a (ϕ, f_s)-graph, where $\{f_s\}$ is isoparametric. Since θ is constant on Σ, so is the ρ-function of any such graph, for $\theta^2 + \rho^2 = 1$. Thus, from Theorem 5-(iii), Σ is trivial. This proves (i).

Let us suppose now that θ vanishes on Σ. Then, from (55), $\lambda_1 \equiv 0$. In this case, we have from (16) that

$$
\begin{aligned}
(n-1)(\beta + \alpha) + \Lambda &= 0 \\
\lambda_i^2 - H\lambda_i + \beta + (n-1)\alpha + \Lambda &= 0,
\end{aligned}
\tag{56}
$$

where, by abuse of notation, we write $\beta \circ \xi = \beta$ and $\alpha \circ \xi = \alpha$.

Considering the first of the above equations and the definitions of α and β given in (7), we conclude that ω must satisfy the following ODE

$$\frac{\omega''}{\omega} = -\frac{\Lambda}{n-1}.$$

A first integration then gives, for some constant c,

$$(\omega')^2 + \frac{\Lambda}{n-1}\omega^2 + c = 0,
\tag{57}$$

which, together with (7), yields

$$\alpha = -\frac{\Lambda}{n-1} - \frac{c+\epsilon}{\omega^2} \quad \text{and} \quad \beta = \frac{c+\epsilon}{\omega^2}.
\tag{58}$$

If $c + \epsilon = 0$, then $\beta = 0$ and we are in the position of Theorem 3-(ii), in which case Σ has constant sectional curvature and its shape operator has rank at most 1.

If $c + \epsilon \neq 0$, we have from (58) that β does not vanish identically on I, so that $I \times_\omega \mathbb{Q}_\epsilon^n$ has nonconstant sectional curvature. Also, since $\theta \equiv 0$, we have $T = \partial_t$, which gives that Σ is a cylinder over a hypersurface Σ_0 of $\mathbb{Q}_\epsilon^n$. Denoting by λ_i^0 the i-th principal curvature of Σ_0, we have from equality (15) in Lemma 1 that $\lambda_i^0 = -\omega(t)\lambda_i$. In this case, the second equation in (56) becomes

$$(\lambda_i^0)^2 - H^0\lambda_i^0 - (n-2)(c+\epsilon) = 0,
\tag{59}$$

where H^0 is the mean curvature of Σ_0. Therefore, at each point of Σ_0, we have at most two distinct principal curvatures. Let us call them λ^0 and μ^0.

Suppose $c + \epsilon < 0$. If $\lambda^0 \neq \mu^0$ at some point, then

$$(k-1)\lambda^0 + (n-k-2)\mu^0 = 0$$

$$\lambda^0 \mu^0 = -(n-2)(c+\epsilon). \tag{60}$$

Then λ_0 and μ_0 do not vanish and have the same sign. The first equation of (60) implies that $k - 1 = n - k - 2 = 0$ and, therefore, $n = 3$. Since we are supposing $n > 3$, we conclude that all points of Σ_0 are umbilical, which implies that Σ has constant sectional curvature.

If $c + \epsilon > 0$, the roots of (59) are necessarily distinct, which implies that $\lambda^0 \neq \mu^0$ everywhere on Σ_0. Moreover, from (60), λ^0 and μ^0 are constant functions. In this case, Cartan's identity $\lambda^0 \mu^0 = -\epsilon$ implies that $\epsilon = 1$. Otherwise, again by the first equation of (60), we would have $n = 3$.

Therefore, denoting by $\mathbb{S}^k(c)$ the k-dimensional sphere of constant sectional curvature c, it follows from [14, Theorems 2.5, 3.4] that, for some $k \in \{1, \ldots, n-2\}$, Σ_0 is congruent to the standard Riemannian product $\mathbb{S}^k(c_1) \times \mathbb{S}^{n-1-k}(c_2)$, where

$$c_1 = \frac{n-3}{k-1} \quad \text{and} \quad c_2 = \frac{n-3}{n-k-2}. \tag{61}$$

This finishes our proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2. It follows from the proof of Theorem 6 that, except for the case (ii)-(a), the Einstein hypersurface Σ in its statement is necessarily ideal.

In the next proof, we shall consider the fact that the isoparametric hypersurfaces of $\mathbb{Q}_\epsilon^n$ having at most two distinct principal curvatures are completely classified. They are the totally umbilical hypersurfaces, in the case of a single principal curvature, and the tubes around totally geodesic submanifolds, in the case of two distinct principal curvatures. In Euclidean space, for instance, they constitute hyperplanes, geodesic spheres, and generalized cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k-1}$, $k \in \{1, \ldots, n-2\}$ (see [3,7] for more details).

Proof of Theorem 7. Keeping the notation of the proof of Theorem 4, we have that Σ has at most three distinct principal curvatures, λ_1, λ and μ. Since $(\beta \circ \xi)T$ never vanishes on Σ, the principal curvature λ_1 cannot satisfy both equations in Lemma 3. Thus, $\lambda \neq \lambda_1 \neq \mu$, i.e., λ_1 has multiplicity one, so that Σ has at least two distinct principal curvatures. Let us define the set

$$C := \{x \in \Sigma \,;\, \lambda(x) = \mu(x)\}.$$

If $\Sigma = C$, we are done. Analogously, if $C = \emptyset$, we have that Σ has three distinct principal curvatures everywhere. Also, since $\{T\}^\perp$ is invariant by the shape operator A of Σ, and λ and μ are the (distinct) eigenvalues of $A|_{\{T\}^\perp}$, we have from [14, Proposition 2.2] that both λ and μ have constant multiplicity.

Now, suppose that C and $\Sigma - C$ are both nonempty and let x be a boundary point of $\Sigma - C$. Since C is closed, one has $x \in C$, so that the vertical section $\Sigma_t \ni x$ is umbilical at x (by (15) in Lemma 1). Furthermore, there is no open set Σ' of Σ containing x such that θ vanishes on Σ'. Indeed, assuming otherwise, we have that Σ' correspond to one of the cases (b) or (c) of Theorem

6-(ii) (see Remark 2). However, in any of them, $\lambda = \mu$ or $\lambda \neq \mu$ everywhere on Σ', which contradicts the fact that x is a boundary point of $\Sigma - C$.

It follows from the above considerations that, in any open set Σ' of Σ containing x, there exists $y \in (\Sigma - C) \cap \Sigma'$ such that $\theta(y) \neq 0$. Then, from Theorem 5, there exists a local (ϕ, f_s)-graph $\Sigma(y) \ni y$ in $\Sigma - C$ whose parallel family $\{f_s\}$ is isoparametric in $\mathbb{Q}_\epsilon^n$. In this case, each parallel f_s has precisely two distinct principal curvatures everywhere, so that they are tubes around totally geodesic submanifolds.

Now, we can choose y as above in such a way that x is a boundary point of $\Sigma(y)$. Then, denoting by π the vertical projection of $I \times \mathbb{Q}_\epsilon^n$ over $\mathbb{Q}_\epsilon^n$, one has that an open neighborhood $U \ni \pi(x)$ in $\pi(\Sigma_t) \subset \mathbb{Q}_\epsilon^n$ is the limit set of a family of parallel open subsets of tubes of the family $\{f_s\}$. In this case, it is easily seen that U itself must be part of a tube of $\{f_s\}$. However, U is umbilical at $\pi(x)$, and tubes have no umbilical points. This contradiction shows that either C or $\Sigma - C$ is empty, proving the first part of the theorem. As for the last part, just apply Theorem 4. $\qquad\square$

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