Weingarten flows in Riemannian manifolds

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Abstract Given orientable Riemannian manifolds M^n and $\overline{M}^{n+1}$, we study flows $F_t : M^n \to \overline{M}^{n+1}$, called Weingarten flows, in which the hypersurfaces $F_t(M)$ evolve in the direction of their normal vectors with speed given by a function W of their principal curvatures, called a Weingarten function, which is homogeneous, monotonic increasing with respect to any of its variables, and positive on the positive cone. We obtain existence results for flows with isoparametric initial data, in which the hypersurfaces $F_t : M^n \to \overline{M}^{n+1}$ are all parallel, and $\overline{M}^{n+1}$ is either a simply connected space form or a rank-one symmetric space of noncompact type. We prove that the avoidance principle holds for Weingarten flows defined by odd Weingarten functions, and also that such flows are embedding preserving.

1. Introduction

Given an open set $\Gamma \subset \mathbb{R}^n$ containing $\Gamma_+ := \{(k_1, \dots, k_n); k_i > 0\}$, we say that $W = W(k_1, \dots, k_n) \in C^\infty(\Gamma)$ is a *Weingarten function* if it is symmetric, homogeneous, monotonic increasing with respect to any of its variables, and positive on Γ_+. For a hypersurface $f : M^n \to \overline{M}^{n+1}$ (M^n and $\overline{M}^{n+1}$ are arbitrary orientable Riemannian manifolds), denote by $k_1, \dots, k_n$ its principal curvature functions. Assuming that $(k_1(p), \dots, k_n(p)) \in \Gamma$ for all $p \in M$, we define the *Weingarten function* W_f of f associated to W as

$$W_f(p) := W\big(k_1(p), \dots, k_n(p)\big), \quad p \in M.$$

If W_f is constant on M, we say that f is a *W-hypersurface*.

The higher-order mean curvatures H_r, $1 \le r \le n$, and the squared norm of the second fundamental form $\|A\|^2$ are distinguished examples of Weingarten functions. They are defined as

$$H_r = \sum_{i_1 < \dots < i_r} k_{i_1} \dots k_{i_r} \qquad \text{and} \qquad \|A\|^2 = \sum_{i=1}^{n} k_i^2.$$

In this paper, we shall consider the problem of finding a one-parameter family of smooth-oriented immersions $F(\cdot, t) : M^n \to \overline{M}^{n+1}$, $t \in [0, T)$, which, for a given Weingarten function $W \in C^\infty(\Gamma)$, satisfy the evolution equation:

$$(1) \qquad \begin{cases} \frac{\partial F}{\partial t}(p,t) = W(p,t)N(p,t) \\ F(p,0) = f(p), \end{cases}$$

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where $N(p,t)$ is the unit normal to the hypersurface $F_t := F(\cdot,t)$, and $W(\cdot,t) = W_{F_t}$ is the Weingarten function of F_t associated to W. We shall call such a family of immersions a *Weingarten flow* (or a *W-flow*, in order to specify the function W) in $\overline{M}^{n+1}$ with *initial data f*.

Huisken and Polden [7] have established existence of short time solutions to (1). Here, we shall seek solutions such that the immersions $F_t : M^n \to \overline{M}^{n+1}$ are all parallel to the initial data f; that is,

$$F_t(p) = \exp_{f(p)}\big(\varphi(t)N(p)\big), \quad (p,t) \in M \times [0,T), \tag{2}$$

where exp stands for the exponential map of $\overline{M}^{n+1}$, $\varphi \in C^\infty[0,T)$ satisfies $\varphi(0) = 0$, and N is the unit normal to f. We call F_t a *parallel φ-flow* and choose

$$N(p,t) = d\exp_{f(p)}\big(\varphi(t)N(p)\big)N(p) \tag{3}$$

as the unit normal field of F_t.

As we shall see, if a parallel φ-flow is a solution to (1), then each immersion $F_t : M^n \to \overline{M}^{n+1}$ is necessarily a W-hypersurface. This fact leads us to consider parallel families in $\overline{M}^{n+1}$ such that, on each hypersurface of the family, the principal curvatures are constant functions. In particular, such families are isoparametric.

Recall that a family of parallel hypersurfaces in a Riemannian manifold is called *isoparametric* if each of them has constant mean curvature. In this case, each hypersurface of the family is also called isoparametric. As proved by Cartan, a hypersurface of a space form is isoparametric if and only if its principal curvatures are constant functions. Despite the existence of manifolds admitting isoparametric hypersurfaces with nonconstant principal curvatures, we shall abuse the terminology and call *isoparametric* only the ones having constant principal curvatures. In this context, the simply connected space forms $\mathbb{Q}_\epsilon^{n+1}$ of constant sectional curvature $\epsilon \in \{0,1,-1\}$, as well as the rank-one symmetric spaces of noncompact type (i.e., the hyperbolic spaces $\mathbb{H}_\mathbb{F}^m$), are natural sources of parallel W-flows since these spaces have many families of isoparametric hypersurfaces (see Section 2.1 for details).

Our first main result, as stated below, concerns parallel Weingarten flows in space forms of nonpositive curvature.

THEOREM 1
For $\epsilon \in \{0,-1\}$, let $f : M^n \to \mathbb{Q}_\epsilon^{n+1}$ be a complete non-totally geodesic isoparametric hypersurface of $\mathbb{Q}_\epsilon^{n+1}$, and let $W \in C^\infty(\Gamma)$ be a Weingarten function. Then there exists a parallel φ-flow F_t defined on a maximal interval $[0,T)$, $T \leq +\infty$, which is a solution to (1), and has the following properties, according to the isoparametric type of $f(M)$:

(i) *If $f(M) \subset \mathbb{H}^{n+1}$ is a horosphere, then $T = +\infty$ and $\{F_t(M), t \in (0,+\infty)\}$ is a family of horospheres in $\mathbb{H}^{n+1}$ which foliates the open horoball bounded by $f(M)$.*
(ii) *If $f(M) \subset \mathbb{H}^{n+1}$ is an equidistant hypersurface to a totally geodesic hyperplane $\Pi \subset \mathbb{H}^{n+1}$, then $T = +\infty$ and $F_t(M) \to \Pi$ as $t \to +\infty$.*

(iii) *If $f(M) \subset \mathbb{Q}_\epsilon^{n+1}$ is either a geodesic sphere or a generalized cylinder, then $T < +\infty$, $\varphi(T)$ is the focal distance of $f(M)$, and $F_t(M)$ collapses into the focal set of $f(M)$ at $t = T$.*

We also consider W-flows in $\mathbb{S}^{n+1}$ from its isoparametric hypersurfaces. It was established by Münzner [10] that the focal set of each family $\mathscr{F}$ of complete isoparametric hypersurfaces of $\mathbb{S}^{n+1}$ has two (and only two) well determined components, $\mathscr{F}_-$ and $\mathscr{F}_+$. In this setting, we shall say that such a family is *positively oriented* if the normal vectors of its hypersurfaces "point toward" $\mathscr{F}_+$ (see Section 2.1 for a precise definition). Then we obtain the following result ($g \in \{1, 2, 3, 4, 6\}$ is the number of distinct principal curvatures of the elements of $\mathscr{F}$).

THEOREM 2
Let $\mathscr{F} = \{f_\tau : M \to \mathbb{S}^{n+1}; \tau \in (0, \pi/g)\}$ be a family of positively oriented isoparametric hypersurfaces of $\mathbb{S}^{n+1}$, and let $W \in C^\infty(\Gamma)$ be a Weingarten function such that W_{f_τ} is well defined for all $f_\tau \in \mathscr{F}$. Given $\tau_0 \in (0, \pi/g)$, assume that $W_{f_{\tau_0}} > 0$ (resp. $W_{f_{\tau_0}} < 0$) and that the function $\tau \mapsto W_{f_{\tau_0 - \tau}}$ (resp. $\tau \mapsto W_{f_{\tau_0 + \tau}}$) is increasing (resp. decreasing) on $[0, \tau_0)$ (resp. on $[0, \pi/g - \tau_0)$). Under these conditions, the maximal parallel φ-flow solution $F_t = f_{\tau_0 - \varphi(t)}$ to (1) with initial data $F_0 = f_{\tau_0}$ collapses into the focal set $\mathscr{F}_+$ (resp. $\mathscr{F}_-$) at $t = \varphi^{-1}(\tau_0)$ (resp. $t = \varphi^{-1}(\tau_0 - \pi/g)$).

It should be mentioned that Theorems 1 and 2 constitute extensions of the main results of [11], where the authors addressed mean curvature flows by parallel hypersurfaces in $\mathbb{Q}_\epsilon^{n+1}$.

By considering the classical isoparametric hypersurfaces of the hyperbolic spaces $\mathbb{H}_\mathbb{F}^m$ (namely, geodesic spheres and horospheres), we establish the following theorem.

THEOREM 3
Let $f : M^n \to \mathbb{H}_\mathbb{F}^m$ be either a horosphere or a geodesic sphere of $\mathbb{H}_\mathbb{F}^m$, and let $W \in C^\infty(\Gamma)$ be a Weingarten function. Then there exists a parallel φ-flow F_t defined on a maximal interval $[0, T)$, $T \leq +\infty$, which is a solution to (1). In addition, the following hold:

(i) *If $f(M)$ is a horosphere, then $T = +\infty$ and $\{F_t(M), t \in (0, +\infty)\}$ is a family of horospheres in $\mathbb{H}_\mathbb{F}^m$ which foliates the open horoball bounded by $f(M)$.*
(ii) *If $f(M)$ is a geodesic sphere, then $T < +\infty$, $\varphi(T)$ is the radius of $f(M)$, and $F_t(M)$ collapses into the center of $f(M)$ at $t = T$.*

Our intent in Theorem 3 is to obtain a unified result—that is, one that would be valid for all hyperbolic spaces $\mathbb{H}_\mathbb{F}^m$. Nevertheless, a similar result could be obtained by considering all isoparametric families of some specific hyperbolic space. In fact, this was done in Theorem 1 for the real hyperbolic space $\mathbb{H}^{n+1}$. On this matter, it could be interesting to explore the results in [1], where the authors, among other accomplishments, determine the principal curvatures of all homogeneous hypersurfaces of the complex hyperbolic space. We remark that homogeneous hypersurfaces are isoparametric and have constant principal curvatures.

An important property shared by many kinds of flows in Euclidean space is the *avoidance principle*, which essentially says that two flows with disjoint initial data remain disjoint until one of them colapses. Here, by means of a result by Hamilton [6], we establish an avoidance principle for W-flows $F_t : M^n \to \overline{M}^{n+1}$ whose Weingarten function $W \in C^\infty(\Gamma)$ is odd. Setting $\mathbf{k} := (k_1, \dots, k_n)$, this means that W admits an extension to $-\Gamma := \{-\mathbf{k}; \mathbf{k} \in \Gamma\}$ which satisfies $W(-\mathbf{k}) = -W(\mathbf{k})$. For instance, as one can easily check, for r odd, the mean curvatures H_r are all odd functions. In our setting, it is also required that the flow takes place in a strongly convex set of the ambient manifold $\overline{M}^{n+1}$. (Recall that a set $\Omega \subset \overline{M}^{n+1}$ is called *strongly convex* if any two points of Ω can be joined by a unique geodesic of $\overline{M}^{n+1}$ which is entirely contained in Ω.)

THEOREM 4 (Avoidance principle)
Let M_1^n, M_2^n, and $\overline{M}^{n+1}$ be complete connected Riemannian manifolds, being M_2^n compact. Assume that $W \in C^\infty(\Gamma)$ is an odd Weingarten function and that

$$F^i : M_i^n \times [0, T) \to \Omega \subset \overline{M}^{n+1}, \quad i = 1, 2$$

are W-flows, where Ω is a strongly convex open set of $\overline{M}^{n+1}$. Under these conditions, we have that the function

$$D(t) := \operatorname{dist}^2\big(F_t^1(M_1), F_t^2(M_2)\big), \quad t \in [0, T)$$

is not decreasing. In particular, if $F_0^1(M_1)$ and $F_0^2(M_2)$ are disjoint, then $F_t^1(M_1)$ and $F_t^2(M_2)$ are disjoint for all $t \in [0, T)$.

As a consequence of the avoidance principle, if $\overline{M}^{n+1}$ is either a space form $\mathbb{Q}_\epsilon^{n+1}$ or a hyperbolic space $\mathbb{H}_\mathbb{F}^m$, then a W-flow $F_t : M^n \to \overline{M}^{n+1}$ of a compact manifold M collapses in a finite time T, provided that W is odd or F_t is an embedding for all $t \in [0, T)$ (see Corollary 5 in Section 3).

In our final result, we show that Weingarten flows defined by odd Weingarten functions preserve embeddedness.

THEOREM 5
Let $\overline{M}^{n+1}$ be a complete connected Riemannian manifold. Assume that $W \in C^\infty(\Gamma)$ is an odd Weingarten function and that

$$F : M^n \times [0, T) \to \overline{M}^{n+1}$$

is a W-flow of a compact connected Riemannian manifold M. Under these conditions, if the initial data F_0 is an embedding, then F_t is an embedding for all $t \in [0, T)$.

The paper is organized as follows. In Section 2, we establish general facts on W-flows by parallel hypersurfaces and present the proofs of Theorems 1–3. We also apply these results to determine the collapsing time of some W-flows in $\mathbb{Q}_\epsilon^{n+1}$ and $\mathbb{H}_\mathbb{F}^m$ as well. In Section 3, we provide the proofs of Theorems 4 and 5.

2. W-flows by parallel hypersurfaces

The following result gives us a way of obtaining Weingarten flows by parallel hypersurfaces. An interesting property of such a flow is that its hypersurfaces are all W-hypersurfaces.

PROPOSITION 1

Given a Weingarten function $W \in C^{\infty}(\Gamma)$, let F_t be a parallel φ-flow as in (2), and assume that W_{F_t} is well defined for all $t \in [0,T)$. Then F_t is a solution to (1) with initial data $f = F_0$ if and only if the function φ satisfies

$$\varphi'(t) = W(p,t) \quad \forall (p,t) \in M \times [0,T). \tag{4}$$

If so, $F_t : M^n \to \overline{M}^{n+1}$ is a W-hypersurface for all $t \in [0,T)$.

Proof

From (2), we have that

$$\frac{\partial F}{\partial t}(p,t) = d\exp_p\big(\varphi(t)N(p)\big)\varphi'(t)N(p) = \varphi'(t)N(p,t).$$

This, together with (3), gives that $F_t = F(\cdot,t)$ satisfies (1) if and only if φ satisfies (4). In particular, if this equality holds, the Weingarten function W_{F_t} is constant on M (possibly depending on t)—that is, F_t is a W-hypersurface of $\overline{M}$. □

As an immediate consequence of Proposition 1, we have the following corollary.

COROLLARY 1

Given a Weingarten function $W \in C^{\infty}(\Gamma)$, let us suppose that

$$\mathcal{F} := \{f_\tau : M^n \to \overline{M}^{n+1}; \tau \in (-\delta,\delta)\} \tag{5}$$

is a family of parallel W-hypersurfaces of $\overline{M}$ defined by $f_\tau(p) = \exp_p(\tau N(p))$, where N is the unit normal to $f = f_0$. Then, writing $W(\tau) = W_{f_\tau}$, we have that the solution $\tau = \varphi(t)$ of the initial value problem

$$\begin{cases} \tau' = W(\tau) \\ \tau(0) = 0 \end{cases} \tag{6}$$

determines a parallel φ-flow solution to (1).

As we pointed out in the introduction, the fact that hypersurfaces of parallel W-flows are W-hypersurfaces suggests the consideration of isoparametric hypersurfaces. We emphasize that here, by abuse of terminology, a one-parameter family $f_\tau : M^n \to \overline{M}^{n+1}$ of parallel hypersurfaces is called *isoparametric* if, for each τ, any principal curvature function k_i of f_τ is constant on M (possibly depending on τ). In this case, each hypersurface f_τ is also called *isoparametric*.

Given a Weingarten function $W \in C^{\infty}(\Gamma)$, it is clear that any isoparametric hypersurface $f : M^n \to \overline{M}^{n+1}$ is a W-hypersurface, provided that W_f is well defined. Therefore, in view of Corollary 1, we have the following result.

COROLLARY 2

Suppose that $f : M^n \to \overline{M}^{n+1}$ is an isoparametric hypersurface. Then, for any Weingarten function $W \in C^\infty(\Gamma)$ for which W_f is well defined, there exists a unique solution to (1) by parallel hypersurfaces with initial data f.

2.1. Parallel W-flows in space forms

Let us apply the results so far obtained to study W-flows in the simply connected space forms $\mathbb{Q}_\epsilon^{n+1}$. In view of Corollary 2, we shall consider the isoparametric hypersurfaces of these spaces. (For details and proofs on this subject we refer to [4].)

For $\epsilon \leq 0$, the complete isoparametric hypersurfaces of $\mathbb{Q}_\epsilon^{n+1}$ are totally classified. They are as follows:

(i) the totally geodesic hyperplanes $\mathbb{Q}_\epsilon^n \subset \mathbb{Q}_\epsilon^{n+1}$
(ii) the geodesic spheres
(iii) the generalized cylinders $\mathbb{Q}_\epsilon^{n-k} \times \mathbb{S}^k$, where $\mathbb{Q}_\epsilon^{n-k}$ is a totally geodesic submanifold of $\mathbb{Q}_\epsilon^{n+1}$ of dimension $n-k<n$ and $\mathbb{S}^k$ is the k-dimensional unit sphere
(iv) the horospheres of $\mathbb{H}^{n+1}$
(v) the equidistant hypersurfaces to totally geodesic hyperplanes of $\mathbb{H}^{n+1}$

In fact, for $\epsilon \leq 0$, any isoparametric hypersurface of $\mathbb{Q}_\epsilon^{n+1}$ is necessarily an open set of one of the complete hypersurfaces listed above.

We point out that, in the cases (ii) and (iii), the isoparametric hypersurfaces have focal points. More specifically, any geodesic sphere has a unique focal point, which is its center, and the focal set of a generalized cylinder $\mathbb{Q}_\epsilon^{n-k} \times \mathbb{S}^k$ is the totally geodesic submanifold $\mathbb{Q}_\epsilon^{n-k}$. In such cases, we shall take the focal distance as the parameter for a family of isoparametric hypersurfaces; that is, if

$$\mathcal{F} = \{f_\tau : M^n \to \mathbb{Q}_\epsilon^{n+1}; \tau \in I \subset \mathbb{R}\}$$

is such a family, then τ is the distance from $f_\tau(M)$ to its focal set. For instance, if $\mathcal{F}$ is a family of concentric geodesic spheres, then $M = \mathbb{S}^n$, $I = [0, +\infty)$ and $\tau > 0$ is the radius of $f_\tau(\mathbb{S}^n)$.

We also observe that all of the above isoparametric hypersurfaces are connected, orientable, properly embedded, and convex. (By *convex*, we mean that with the inward orientation, which is the one we shall adopt here, the principal curvatures of the hypersurface are all nonnegative.) In addition, the isoparametric hypersurfaces in (ii), (iv), and (v) are all *strictly convex*—that is, all their principal curvatures are positive everywhere.

Proof of Theorem 1

First, let us write

$$\mathcal{F} = \{f_\tau : M^n \to \mathbb{Q}_\epsilon^{n+1}; \tau \in I \subset \mathbb{R}\}$$

for the isoparametric family of complete hypersurfaces of $\mathbb{Q}_\epsilon^{n+1}$ (defined in a maximal interval $I \subset \mathbb{R}$) such that $f = f_{\tau_0}$, $\tau_0 \in I$. From the convexity of the hypersurfaces f_τ, and from the positivity of W on Γ_+, we have that $W_{f_\tau} \geq 0$ for all $\tau \in I$.

If $\mathscr{F}$ is a family of horospheres, then $I = \mathbb{R}$ and, for any $\tau \in \mathbb{R}$, all principal curvatures of f_τ are equal to 1, which implies that $W_{f_\tau} = W$ is a positive constant independent of τ. Hence, by Corollary 1, the function $\varphi(t) = Wt$, $t \in [0, +\infty)$, determines a φ-flow F_t which is a solution to (1) with initial data f_{τ_0}; namely,

$$F_t = f_{Wt+\tau_0}, t \in [0, +\infty).$$

Clearly, for all $t > 0$, $F_t(M)$ is a horosphere of $\mathbb{H}^{n+1}$ contained in the open horoball bounded by $f(M)$. This proves (i).

Assume now that $\mathscr{F}$ is a family of equidistant hypersurfaces to a totally geodesic hyperplane $\Pi \subset \mathbb{H}^{n+1}$. In this case, $I = \mathbb{R}$, and the parameter $\tau > 0$ is the distance from $f_\tau(M)$ to Π. We can assume, without loss of generality, that $\tau_0 > 0$. Let $\varphi : [0, T) \to \mathbb{R}$ be the solution of (6) defined in a maximal interval $[0, T)$. Then

$$F_t = f_{\tau_0 - \varphi(t)}, \quad t \in [0, T)$$

is a solution to (1) satisfying $F_0 = f_{\tau_0}$. Assume, by contradiction, that $T < +\infty$. If $\varphi(T) = \tau_0$, then the flow F_t can be extended beyond T just by setting $F_t = f_0$ for $t \geq T$ (since, by the homogeneity of W, $W(0, 0, \dots, 0) = 0$), contradicting the maximality of T. Analogously, if $\varphi(T) < \tau_0$, we have that $F_T = f_{\tau_0 - \varphi(T)}$ is well defined, so that we can extend the flow F_t beyond T—again a contradiction. Therefore, $T = +\infty$.

If $\varphi(t_0) = \tau_0$ for some $t_0 \in (0, +\infty)$, then $F_t(M) = \Pi$ for all $t \geq t_0$. Hence, we can assume that φ is bounded above by τ_0. In this case, since $F_t(M)$ moves toward Π, the principal curvatures of F_t are positive decreasing functions of t. This, together with the monotonicity property of W, gives that the function $W(\varphi(t))$ $(= W_{F_t} = W_{f_{\tau_0 - \varphi(t)}})$ decreases as $t \to +\infty$. Since $\varphi'(t) = W(\varphi(t))$, we conclude that $\varphi''(t) < 0$—that is, φ is positive, increasing, concave, and bounded on $[0, +\infty)$. These properties clearly imply that $\varphi'(t) \to 0$ as $t \to +\infty$. Therefore,

$$W_{f_0} = 0 = \lim_{t \to +\infty} \varphi'(t) = \lim_{t \to +\infty} W\big(\varphi(t)\big) = \lim_{t \to +\infty} W_{f_{\tau_0 - \varphi(t)}},$$

which yields $\lim_{t \to +\infty} \varphi(t) = \tau_0$. Consequently, $F_t \to f_0$ as $t \to +\infty$, which shows assertion (ii).

Finally, let us suppose that $\mathscr{F}$ is a family of concentric geodesic spheres of $\mathbb{Q}_\epsilon^{n+1}$ (the argument for generalized cylinders is analogous). In this setting, let $\varphi : [0, T) \to \mathbb{R}$ be the solution of (6), so that $F_t = f_{\tau_0 - \varphi(t)}$ is the solution to (1) satisfying $F_0 = f_{\tau_0}$. Since $F_t(M)$ flows toward the center of the spheres $f_\tau(M)$, we have that $\varphi(t) < \tau_0$ for all $t \in [0, T)$, and also that $W(\varphi(t)) = \varphi'(t)$ is a positive increasing function of t. Thus, $\varphi'' > 0$—that is, φ is bounded, increasing, and strictly convex, which clearly implies that $T < \infty$. Moreover, we must have $\varphi(T) = \tau_0$. Otherwise, arguing as in the preceding paragraph, we derive a contradiction by extending φ beyond T. This completes the proof of (iii), and so of the theorem. $\square$

Let us consider now the isoparametric hypersurfaces of $\mathbb{S}^{n+1}$. A well-known result asserts that any such hypersurface has exactly g distinct principal curvatures, where $g \in \{1, 2, 3, 4, 6\}$. The case $g = 1$, for instance, corresponds to the geodesic spheres of $\mathbb{S}^{n+1}$. Rather than using the classification theorems for isoparametric hypersurfaces of

$\mathbb{S}^{n+1}$, we shall consider their characterization in terms of level sets of homogeneous polynomials, as done by Münzner [10] (see also [4]).

To clearer, let $f : M \to \mathbb{S}^{n+1}$ be a complete isoparametric hypersurface with g distinct principal curvatures. Münzner's result asserts that $f(M)$ is the intersection of $\mathbb{S}^{n+1}$ with a level set $P^{-1}(c)$, $c \in (-1,1)$, of a homogeneous polynomial function $P : \mathbb{R}^{n+2} \to \mathbb{R}$ of degree g satisfying certain differential equations. Distinct level sets of P are necessarily parallel in $\mathbb{S}^{n+1}$, and the focal set of this parallel family has precisely two connected components, which are the intersections of $\mathbb{S}^{n+1}$ with $P^{-1}(-1)$ and $P^{-1}(1)$. In addition, given $p \in M$, if we write $\gamma\colon (0,\pi/g) \to \mathbb{S}^{n+1}$ for the normalized geodesic from $P^{-1}(1)$ to $P^{-1}(-1)$ which is orthogonal to f at $p = \gamma(\tau)$ and set the *positive orientation* $N(p) = -\gamma'(\tau)$ for f, then its g distinct principal curvatures are given by

$$k_i = \cot\Big(\tau + (i-1)\frac{\pi}{g}\Big), \quad 1 \le i \le g. \tag{7}$$

In this setting, $\tau \in (0,\pi/g)$ is the focal distance from $f(M)$ to $P^{-1}(1)$. Also, all principal curvatures of f increase as τ decreases to 0 and decrease as τ increases to π/g. We shall denote the multiplicity of k_i by m_i.

Summarizing, we have that any isoparametric hypersurface of $\mathbb{S}^{n+1}$ with g distinct principal curvatures is an open subset of an element of a family

$$\mathcal{F} = \{f_\tau\colon M \to \mathbb{S}^{n+1}; \tau \in (0,\pi/g)\}$$

of complete isoparametric hypersurfaces such that $f_\tau(M)$ is at a distance τ from the focal component $\mathcal{F}_+ := P^{-1}(1)$. For γ as above, the family $\mathcal{F}$ is said to be *positively oriented* if the unit normal N of any $f_\tau \in \mathcal{F}$ at $p = \gamma(\tau)$ is $N(p) = -\gamma'(\tau)$.

Proof of Theorem 2
Suppose that, for some $\tau_0 \in (0,\pi/g)$, $W_{f_{\tau_0}} > 0$. In this case, if the function $\tau \mapsto W_{f_{\tau_0-\tau}}$ is increasing on $[0,\tau_0)$, the φ-flow

$$F_t = f_{t-\varphi(t)}, \qquad \varphi(0) = 0, \qquad \varphi'(t) = W_{f_{t-\varphi(t)}},$$

moves toward $\mathcal{F}_+$ with increasing velocity. Hence, arguing as in the proof of Theorem 1, case (iii), we conclude that $F_t(M)$ collapses into $\mathcal{F}_+$ at $t = \varphi^{-1}(\tau_0)$.

Analogously, if $W_{f_{\tau_0}} < 0$ and the function $\tau \mapsto W_{f_{\tau_0+\tau}}$ is decreasing on the interval $[0,\pi/g - \tau_0)$, then $F_t(M)$ collapses into the focal component $\mathcal{F}_- := P^{-1}(-1)$ at $t = \varphi^{-1}(\tau_0 - \pi/g)$. □

Let us see now that Theorem 2 applies when W is either the higher-order mean curvature H_r or the squared norm of the second fundamental form $\|A\|^2$.

Let $\mathcal{F}$ be as in Theorem 2. Then, in any open interval $(0,\tau_0)$, $0 < \tau_0 < \pi/g$, we have that $k_1^\tau = \cot\tau$ is unbounded, whereas $k_i^\tau = \cot(\tau + (i-1)\pi/g)$, $i = 2,\dots,g$, is bounded. Assuming that the multiplicity m_1 of k_1^τ (which is the same for all τ) satisfies $m_1 \ge r$, where $r \in \{1,\dots,n-1\}$, the rth mean curvature $H_r(\tau)$ of f_τ is given by

$$H_r(\tau) = \binom{m_1}{r}\cot^r\tau + \sum_{i=0}^{r-1}\mu_i(\tau)\cot^i\tau,$$

where the functions μ_i are all bounded in $(0, \tau_0)$. In particular, if τ_0 is sufficiently small, $H_r(\tau_0) > 0$, and the function $\tau \in [0, \tau_0) \mapsto H_r(\tau_0 - \tau)$ is increasing. In the same manner, if r is odd, $m_g \geq r$, and τ_0 is sufficiently close to π/g, then $H_r(\tau_0) < 0$, and the function $\tau \in [0, \pi/g - \tau_0) \mapsto H_r(\tau_0 + \tau)$ is decreasing. Thus, we have the following.

COROLLARY 3

Theorem 2 applies to the Weingarten function $W = H_r$, $1 \leq r \leq n-1$. More precisely, given $\tau_0 \in (0, \pi/g)$, if $m_1 \geq r$ (resp. $m_g \geq r$, r odd), and $H_r(\tau_0) > 0$ (resp. $H_r(\tau_0) < 0$), the maximal parallel φ-flow solution $F_t = f_{\tau_0 - \varphi(t)}$ to H_r-flow with initial data $F_0 = f_{\tau_0}$ collapses into the focal set $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) at $t = \varphi^{-1}(\tau_0)$ (resp. $t = \varphi^{-1}(\tau_0 - \pi/g)$).

Theorem 2 also applies to the norm of the second fundamental form $\|A\|^2$ since $\|A\|^2(\tau_0 - \tau)$ is clearly increasing on $[0, \tau_0)$ for all sufficiently small $\tau_0 \in (0, \pi/g)$.

COROLLARY 4

Let $\mathcal{F}$ be as in Theorem 2. Given a sufficiently small $\tau_0 \in (0, \pi/g)$, the maximal parallel φ-flow solution $F_t = f_{\tau_0 - \varphi(t)}$ to $\|A\|^2$-flow with initial data $F_0 = f_{\tau_0}$ collapses into the focal set $\mathcal{F}_+$ at $t = \varphi^{-1}(\tau_0)$.

Next, we apply the results of this section to determine the collapsing time of some parallel W-flows in $\mathbb{Q}_\epsilon^{n+1}$. For that, we shall consider the trigonometric functions $\cos_\epsilon$ and $\sin_\epsilon$ as defined in Table 1. The functions $\tan_\epsilon$, $\cot_\epsilon$, and $\sec_\epsilon$ are defined accordingly—that is, $\tan_\epsilon = \sin_\epsilon / \cos_\epsilon$, $\cot_\epsilon = \cos_\epsilon / \sin_\epsilon$, and $\sec_\epsilon = 1/\cos_\epsilon$.

EXAMPLE 1 (Parallel $\|A\|^2$-flow in $\mathbb{Q}_\epsilon^{n+1}$ with spherical initial data)
Let

$$f : \mathbb{S}^n \to \mathbb{Q}_\epsilon^{n+1}$$

be a (totally umbilical) strictly convex geodesic sphere of $\mathbb{Q}_\epsilon^{n+1}$ of radius $R > 0$ and principal curvature $k = \cot_\epsilon R$. Set

$$\mathcal{R}_\epsilon =: \begin{cases} \pi/2 & \text{if } \epsilon = 1 \\ +\infty & \text{if } \epsilon \neq 1 \end{cases}$$

and let

$$\mathcal{F} = \left\{ f_\tau : \mathbb{S}^n \to \mathbb{Q}_\epsilon^{n+1}; \tau \in (0, \mathcal{R}_\epsilon) \right\}$$

Table 1. Definition of $\cos_\epsilon$ and $\sin_\epsilon$.

Function	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = -1$
$\cos_\epsilon(s)$	1	$\cos s$	$\cosh s$
$\sin_\epsilon(s)$	s	$\sin s$	$\sinh s$

be the family of parallel geodesic spheres of $\mathbb{Q}_\epsilon^{n+1}$ such that $f_R = f$. By Theorems 1 and 2, for $W = \|A\|^2$, the flow

$$F_t := f_{R-\varphi(t)}, \quad t \in [0, T),$$

where φ satisfies

$$\varphi'(t) = W_{f_{R-\varphi(t)}} = n \cot_\epsilon^2\big(R - \varphi(t)\big), \quad \varphi(0) = 0, \tag{8}$$

and is a solution to (1) which collapses into the center of $f(\mathbb{S}^n)$ at time $T = \varphi^{-1}(R)$. Separating variables in (8), we obtain the equation

$$\tan_\epsilon^2(R-\varphi)\, d\varphi = n\, dt,$$

which yields

$$\begin{aligned} \big(R - \varphi(t)\big)^3 &= R^3 - 3nt \quad (\text{for } \epsilon = 0), \\ \tan_\epsilon\big(R - \varphi(t)\big) + \varphi(t) &= \frac{1}{k} - \epsilon n t \quad (\text{for } \epsilon = \pm 1). \end{aligned} \tag{9}$$

Hence, by making $t = T = \varphi^{-1}(R)$ in (9), one concludes that the collapsing time T for the $\|A\|^2$-flow F_t with initial data f is

$$T = \begin{cases} \frac{R^3}{3n} & (\text{for } \epsilon = 0), \\ \frac{\epsilon(1-kR)}{kn} & (\text{for } \epsilon = \pm 1). \end{cases} \tag{10}$$

EXAMPLE 2 (Parallel H_r-flow in $\mathbb{Q}_\epsilon^{n+1}$ with spherical initial data)
Let f and $\mathcal{F}$ be as in the preceding example. For the H_r-flow, the differential equation for φ is

$$\varphi'(t) = W_{f_{R-\varphi(t)}} = \binom{n}{r} \cot_\epsilon^r\big(R - \varphi(t)\big), \quad \varphi(0) = 0, \tag{11}$$

which separates as

$$\tan_\epsilon^r(R-\varphi)\, d\varphi = \binom{n}{r} dt. \tag{12}$$

For $\epsilon = 0$, the solution φ is given implicitly by

$$\frac{(R-\varphi(t))^{r+1}}{r+1} = \frac{R^{r+1}}{r+1} - \binom{n}{r} t,$$

which yields

$$T = \binom{n}{r}^{-1} \frac{R^{r+1}}{r+1}$$

for the collapsing time of F_t.

For $\epsilon = \pm 1$, integration on the left-hand side of (12) is recurrent. In Table 2, we list the solutions φ and corresponding collapsing times for $r = 1, 2$.

Table 2. Function φ and collapsing time T for spherical parallel H_r-flows.

r	φ	T
1	$\cos_\epsilon(R-\varphi(t)) = e^{\epsilon nt}\cos_\epsilon R$	$\frac{\epsilon}{n}\log(1/\cos_\epsilon R)$
2	$\tan_\epsilon(R-\varphi(t)) + \varphi(t) = \frac{1}{k} - \epsilon\frac{n(n-1)}{2}t$	$\frac{2\epsilon(1-kR)}{kn(n-1)}$

EXAMPLE 3 (Parallel K-flow in $\mathbb{S}^{n+1}$ with nonspherical initial data)
Consider an isoparametric family $\mathscr{F}$ of hypersurfaces $f_\tau : M^n \to \mathbb{S}^{n+1}$, $\tau \in (0, \pi/2)$, with two distinct principal curvatures

$$k_1^\tau = \cot\tau \qquad \text{and} \qquad k_2^\tau = \cot(\tau + \pi/2) = -\tan\tau,$$

whose multiplicities are m_1 and m_2, respectively. By a result due to Cartan, M is homeomorphic to the product $\mathbb{S}^{m_1} \times \mathbb{S}^{m_2}$, and the focal components $\mathscr{F}_-$ and $\mathscr{F}_+$ are isometric to the standard spheres $\mathbb{S}^{m_1}$ and $\mathbb{S}^{m_2}$, respectively. Assuming m_2 even, we have that the Gaussian curvature $K(\tau)$ of f_τ is

$$K(\tau) = \cot^{m_1}(\tau)\tan^{m_2}(\tau),$$

which is clearly a positive function on $(0, \pi/2)$.

If $m_1 = m_2$, then $K = 1$ for all $\tau \in (0, \pi/2)$. In this case, given $\tau_0 \in (0, \pi/2)$, the flow $F_t = f_{\tau_0 - t}$ is a solution to K-flow with initial data f_{τ_0} and collapsing time $T = \tau_0$.

If $m_1 > m_2$, the function $K(\tau_0 - \tau) = \cot^{m_1 - m_2}(\tau_0 - \tau)$ is increasing in $[0, \tau_0)$. So, considering the solution φ of $\tau' = K(\tau)$ such that $\varphi(0) = 0$, we have from Theorem 2 that the flow $F_t = f_{\tau_0 - \varphi(t)}$ collapses into $\mathscr{F}_+$ at $T = \varphi^{-1}(\tau_0)$. In addition, setting $m = m_1 - m_2$, we obtain the function φ and the collapsing time T by integrating $\tan^m(\tau_0 - \varphi)$ with respect to φ, as in the preceding example. For instance, if $m = 1$, the implicit equation for φ and collapsing time T are

$$\cos\big(\tau_0 - \varphi(t)\big) = e^t \cos\tau_0 \qquad \text{and} \qquad T = \log(\sec\tau_0).$$

An analogous reasoning applies if m_2 is odd and $m_1 < m_2$, in which case F_t collapses into $\mathscr{F}_-$ at $T = \varphi^{-1}(\tau_0 - \pi/2)$.

2.2. Parallel W-flows in rank-one symmetric spaces

Let us consider now the rank-one symmetric spaces of noncompact type, which are precisely the hyperbolic spaces described through the four normed division algebras: $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{K}$ (quaternions), and $\mathbb{O}$ (octonions). They are denoted by $\mathbb{H}^m_{\mathbb{R}}$, $\mathbb{H}^m_{\mathbb{C}}$, $\mathbb{H}^m_{\mathbb{K}}$, and $\mathbb{H}^2_{\mathbb{O}}$ and called *real hyperbolic space, complex hyperbolic space, quaternionic hyperbolic space*, and *Cayley hyperbolic plane*, respectively. The real hyperbolic space is $\mathbb{H}^{n+1}$ (i.e., the simply connected space form of constant sectional curvature -1). We shall adopt the unified notation $\mathbb{H}^m_{\mathbb{F}}$ for these hyperbolic spaces, where $m = 2$ for $\mathbb{F} = \mathbb{O}$. The real dimension of $\mathbb{H}^m_{\mathbb{F}}$ is $n + 1 = m\dim\mathbb{F}$. In particular, $\mathbb{H}^2_{\mathbb{O}}$ has dimension $n + 1 = 16$.

We add that any hyperbolic space $\mathbb{H}^m_{\mathbb{F}}$ is a Hadamard manifold. After a suitable scale of its metric, its sectional curvatures vary in the interval $[-1, -1/4]$. Moreover, its geodesic spheres and horospheres are all isoparametric and strictly convex (see [2] for details and proofs).

Proof of Theorem 3
Suppose that $f(M)$ is a geodesic sphere, and set

$$\mathscr{F} = \left\{ f_\tau : \mathbb{S}^n \to \mathbb{H}^m_{\mathbb{F}} ; \tau \in (0, +\infty) \right\} \tag{13}$$

for the isoparametric family of geodesic spheres of $\mathbb{H}^m_{\mathbb{F}}$ such that $f = f_{\tau_0}$ for some $\tau_0 \in (0, +\infty)$. (Recall that the parameter τ is the radius of $f_\tau(\mathbb{S}^n)$.)

The principal curvatures k_i^τ of f_τ with respect to the inward orientation are

$$\begin{aligned} k_1^\tau &= \coth(\tau) \quad \text{with multiplicity } q \\ k_2^\tau &= \frac{1}{2}\coth(\tau/2) \quad \text{with multiplicity } n-q, \end{aligned} \tag{14}$$

where $q = n$ for $\mathbb{H}^{n+1}_{\mathbb{R}}$, $q = 1$ for $\mathbb{H}^m_{\mathbb{C}}$, $q = 3$ for $\mathbb{H}^m_{\mathbb{K}}$, and $q = 7$ for $\mathbb{H}^2_{\mathbb{O}}$ (see, e.g., [4, pp. 353 and 543] and [8]). In particular, we have

$$\lim_{\tau \to 0} k_i^\tau = +\infty, \quad i = 1, 2.$$

From the above considerations (and the monotonicity property of W), just as in the real case, we conclude that the parallel φ-flow with initial data $f = f_{\tau_0}$ collapses to its center at $T = \varphi^{-1}(\tau_0)$.

As we pointed out, the horospheres of any hyperbolic space $\mathbb{H}^m_{\mathbb{F}}$ are isoparametric. In fact, as in the real case, they foliate $\mathbb{H}^m_{\mathbb{F}}$ and have all the same constant principal curvatures (cf. the proposition on p. 88 of [2]). So, any horosphere of $\mathbb{H}^m_{\mathbb{F}}$ moves indefinitely with constant speed under any W-flow. □

In the next example, we calculate the collapsing time of a geodesic sphere of $\mathbb{H}^m_{\mathbb{F}}$ moving under H-flow.

EXAMPLE 4 (Parallel H-flow in $\mathbb{H}^m_{\mathbb{F}}$ with spherical initial data)
Let $\mathscr{F}$ be as in (13). Then the mean curvature $H(\tau)$ of f_τ is

$$H(\tau) = q\coth(\tau) + \frac{n-q}{2}\coth(\tau/2).$$

Given $R \in (0, +\infty)$, by Corollary 1 and Theorem 3, the flow

$$F_t := f_{R-\varphi(t)}, \quad t \in [0, T),$$

where φ satisfies

$$\begin{cases} \varphi'(t) = H(R-\varphi(t)) = q\coth(R-\varphi(t)) + \frac{n-q}{2}\coth((R-\varphi(t))/2) \\ \varphi(0) = 0, \end{cases} \tag{15}$$

is a solution to (1) with initial data f_R which collapses into the center of $f_R(\mathbb{S}^n)$ at time $T = \varphi^{-1}(R)$.

From (15), we obtain the equation

$$\frac{d\varphi}{q\coth(R-\varphi)+((n-q)/2)\coth((R-\varphi)/2)}=dt.$$

Setting $x=e^{R-\varphi}$ and integrating the resulting rational function $f(x)/g(x)$ by means of the identities

- $\int\frac{x\,dx}{ax^2+bx+c}=\frac{1}{2a}\log|ax^2+bx+c|-\frac{b}{2a}\int\frac{dx}{ax^2+bx+c}+C,$
- $\int\frac{dx}{x(ax^2+bx+c)}=\frac{1}{2c}\log|\frac{x^2}{ax^2+bx+c}|-\frac{b}{2c}\int\frac{dx}{ax^2+bx+c}+C,$

we conclude that the solution φ of (15) is given implicitly by

$$\log\Big(\frac{e^{R-\varphi(t)}}{a(e^{2(R-\varphi(t))}+1)+be^{R-\varphi(t)}}\Big)^{1/a}=t+C(R),$$

where $a=(n+q)/2$, $b=n-q$, and

$$C(R)=\log\Big(\frac{e^{R}}{a(e^{2R}+1)+be^{R}}\Big)^{1/a}.$$

Therefore, the collapsing time $T=\varphi^{-1}(R)$ is

$$T=\log\Big(\frac{a(e^{2R}+1)+be^{R}}{2ne^{R}}\Big)^{\frac{1}{a}}.$$

3. Avoidance principle for Weingarten flows

In this section, we prove Theorem 4, which constitutes an avoidance principle for Weingarten flows whose corresponding Weingarten functions are odd, as we mentioned in the introduction. The fundamental property of such a W-flow $F_t\colon M^n\to\overline{M}^{n+1}$ is that it is invariant under change of orientation. Indeed, given $(p,t)\in M\times[0,T)$, writing $k_i=k_i(p,t)$ and $N=N(p,t)$, one has

$$\begin{aligned}W(-k_1,\dots,-k_n)(-N)&=-W(k_1,\dots,k_n)(-N)\\&=W(k_1,\dots,k_n)N\\&=\frac{\partial F}{\partial t}(p,t).\end{aligned}\tag{16}$$

Along the proof of Theorem 4, we shall consider graphs over tangent spaces of hypersurfaces, as described below.

Let $f:M^n\to\overline{M}^{n+1}$ be an oriented hypersurface. Fix $p\in M$, and let $U\subset T_pM$ be an open neighborhood of the zero vector of the tangent space of M at p. Given a function $\phi\in C^\infty(U)$, we call the set (assuming it is well defined)

$$\Sigma_\phi:=\big\{\exp_{f(p)}\big(v+\phi(v)N(p)\big)\in\overline{M};v\in U\big\}$$

the *graph* of ϕ on U. Here, exp denotes the exponential map of $\overline{M}^{n+1}$.

Clearly, Σ_ϕ is an orientable hypersurface of $\overline{M}^{n+1}$. Moreover, it is a well-known fact that, if the zero vector $0\in U$ is a critical point of ϕ, then the Hessian of ϕ at 0 coincides with the second fundamental form of Σ_ϕ at $\overline{p}=\exp_{f(p)}(\phi(0)N(p))\in\Sigma_\phi$.

In this case, we consider in Σ_ϕ the orientation such that the unit normal to Σ_ϕ at $\bar{p}$ is (cf. [3, Theorem 3, p. 198]):

$$N_\phi(\bar{p}) = d\exp_{f(p)}\big(\phi(0)N(p)\big)N(p).$$

Notice that, if $\gamma : [0, L] \to \overline{M}^{n+1}$ is the normalized geodesic from $f(p)$ to $\bar{p}$ satisfying $\gamma'(0) = N(p)$, then $N_\phi(\bar{p}) = \gamma'(L)$.

The following elementary result, which will be useful to us, compares principal curvatures of graphs whose corresponding functions have a common critical point. We adopt the convention of ordering the principal curvatures as $k_1 \le \cdots \le k_n$.

LEMMA 1

With the above notation, assume that $\phi, \mu \in C^\infty(U)$ satisfy $\mu \ge \phi$ on U, and that the null vector $0 \in U$ is a minimum of $\mu - \phi$. Then any principal curvature of Σ_μ at $\overline{p} = \exp_{f(p)}(\mu(0)N(p))$ is greater than or equal to the corresponding principal curvature of Σ_ϕ at $\overline{q} = \exp_{f(p)}(\phi(0)N(p))$.

Proof

Since 0 is a minimum of $\mu - \phi$, we have that the Hessian of $\mu - \phi$ at 0 is positive semi-definite, which implies that the same is true for the operator $A_\mu - A_\phi$, where A_μ and A_ϕ are the shape operators of Σ_μ at $\bar{q}$ and Σ_ϕ at $\bar{p}$, respectively. However, a standard result in linear algebra (see theorem on p. 130 in [5]) asserts the following: If A is self-adjoint and B is positive semi-definite, then the eigenvalues of A do not exceed the corresponding ones of $A + B$. Hence, setting $A = A_\phi$ and $B = A_\mu - A_\phi$, the lemma follows. □

The next result, due to Hamilton [6] (see also [9]), will play a fundamental role in the sequel.

LEMMA 2 (Hamilton's trick)

Let $u\colon M \times [0, T) \to \mathbb{R}$ be a C^1 function with the following property: For each $t_0 \in [0, T)$, there exist $\delta > 0$ and a compact subset $\Omega \subset M - \partial M$ such that, for any $t \in (t_0 - \delta, t_0 + \delta)$, the minimum

$$u_{\min}(t) := \min_{p\in M} u(p, t)$$

is attained (at least) at one point of Ω. Then the function $u_{\min}$ is locally Lipschitz in $(0, T)$ and, for each $t \in (0, T)$ where it is differentiable, one has

$$u'_{\min}(t) = \frac{\partial u}{\partial t}(p_0, t),$$

where $p_0 \in M - \partial M$ is any interior point at which $u(\cdot, t)$ attains its minimum.

Proof of Theorem 4

Since M_2 is compact, for each $t \in (0, T)$, there exists a pair $(p_1, p_2) \in M_1 \times M_2$ (possibly depending on t) such that

$$D(t) = \operatorname{dist}^2\big(F_t^1(p_1), F_t^2(p_2)\big). \tag{17}$$

In addition, dist^2 is smooth on Ω, which implies that the function

$$u(p,q,t) := \mathrm{dist}^2\big(F_t^1(p), F_t^2(q)\big), \quad (p,q,t) \in M_1 \times M_2 \times [0,T)$$

is smooth as well. Thus, Hamilton's trick applies and gives that $D(t) = u_{\min}(t)$ is locally Lipschitz, so that D is differentiable almost everywhere (by Rademacher's theorem). Also, at a differentiable point t_0, the following equality holds:

$$D'(t_0) = \frac{\partial u}{\partial t}(p_1, p_2, t_0), \tag{18}$$

where $(p_1, p_2) \in M_1 \times M_2$ is any pair at which $u(\cdot, t_0)$ attains its minimum. So, it suffices to prove that $D'(t_0) \geq 0$. This is certainly true if $D(t_0) = 0$ (since D is nonnegative), so that we can assume $D(t_0) \neq 0$.

In the above setting, the minimizing normalized geodesic $\gamma_{t_0}\colon [0,L] \to \overline{M}$ joining the points $\bar{p}_1 := F_{t_0}^1(p_1)$ and $\bar{p}_2 := F_{t_0}^2(p_2)$ is orthogonal to both $F_{t_0}^1(M_1)$ (at $\bar{p}_1 = \gamma_{t_0}(0)$) and $F_{t_0}^2(M_2)$ (at $\bar{p}_2 = \gamma_{t_0}(L)$).

Let us denote by Π the tangent space of $F_{t_0}^1(M_1)$ at $\bar{p}_1$. It is easily checked that, for $i = 1,2$, there exists an open neighborhood U of 0 in Π such that, for all t sufficiently close to t_0, in a suitable neighborhood of $\bar{p}_i$ in $\overline{M}$, $F_t^i(M_i)$ is a graph of a function $\phi_t^i \in C^\infty(U)$. In particular, we have $\phi_t^2 > \phi_t^1$ on U. Also, since $\bar{p}_i = \phi_{t_0}^i(0)$, we have that $0 \in U$ is a minimum of $\phi_{t_0}^2 - \phi_{t_0}^1$ on U.

By (16), we can assume that $N_{t_0}^1(p_1) = \gamma'(0)$ and $N_{t_0}^2(p_2) = \gamma'(L)$. In this case, from Lemma 1, no principal curvature of $F_{t_0}^1(M_1)$ at p_1 exceeds the corresponding one of $F_{t_0}^2(M_2)$ at p_2. Therefore, from the monotonicity property of the Weingarten function W, the following inequality holds:

$$W_{F_{t_0}^1}(p_1) \leq W_{F_{t_0}^2}(p_2). \tag{19}$$

Now observe that the gradient of the squared distance function of $\overline{M}$ at the point $(\bar{p}_1, \bar{p}_2) \in \overline{M} \times \overline{M}$ is the vector $\nabla\,\mathrm{dist}^2(\bar{p}_1, \bar{p}_2) = 2\,\mathrm{dist}(\bar{p}_1, \bar{p}_2)(-\gamma_{t_0}'(0), \gamma_{t_0}'(L))$. So,

$$\nabla\,\mathrm{dist}^2(\bar{p}_1, \bar{p}_2) = 2\,\mathrm{dist}(\bar{p}_1, \bar{p}_2)\big(-N_{t_0}^1(p_1), N_{t_0}^2(p_2)\big) \in T_{\bar{p}_1}\overline{M} \times T_{\bar{p}_2}\overline{M}. \tag{20}$$

Putting together identities (17)–(20), and considering the fact that F_t^1 and F_t^2 are both W-flows, we have

$$\begin{aligned} D'(t_0) &= \frac{\partial}{\partial t}\,\mathrm{dist}^2\big(F^1(p_1,t), F^2(p_2,t)\big)|_{t=t_0} \\ &= \Big\langle \nabla\,\mathrm{dist}^2(\bar{p}_1, \bar{p}_2), \Big(\frac{\partial F^1}{\partial t}(p_1,t_0), \frac{\partial F^2}{\partial t}(p_2,t_0)\Big)\Big\rangle_{\overline{M}\times\overline{M}} \\ &= 2\,\mathrm{dist}(\bar{p}_1, \bar{p}_2)\big(-W_{F_{t_0}^1}(p_1) + W_{F_{t_0}^2}(p_2)\big) \geq 0, \end{aligned}$$

as we wished to prove. $\square$

In the above proof, the hypothesis of W being odd allowed us to choose the orientation of the hypersurfaces $F_{t_0}^i$ at p_i, $i = 1,2$, in such a way that their unit normals at these points would coincide with $\gamma'(0)$ and $\gamma'(L)$. In this manner, we could apply Lemma 1 and then obtain the fundamental inequality (19). From this, we conclude that we can drop the assumption on W being odd in the statement of the avoidance principle, as long as we have ensured that the orientations of F_t^i follow this pattern.

For instance, suppose that $F_t : M^n \to \overline{M}^{n+1}$, $t \in [0,T)$, is a W-flow, where M^n is compact, and $\overline{M}^{n+1}$ is either a space form $\mathbb{Q}_\epsilon^{n+1}$ or a hyperbolic space $\mathbb{H}_{\mathbb{F}}^m$. Assume that $F_0(M)$ is contained in an open totally convex ball $B_R \subset \overline{M}^{n+1}$, whose boundary ∂B_R is a strictly convex geodesic sphere of $\overline{M}^{n+1}$ (i.e., $0 < R < \pi/2$ for $\overline{M}^{n+1} = \mathbb{S}^{n+1}$). In this setting, considering the parallel flow $P_t\colon\mathbb{S}^n \to \overline{M}^{n+1}$ with initial data $P_0(\mathbb{S}^n) = \partial B_R$ and inward orientation, and assuming that F_t is an embedding with the inward orientation for all $t \in [0,T)$, we have that the normals at the points minimizing the distance between $F_t(M)$ and $P_t(M)$ coincide with the tangent vectors to the minimizing geodesic joining them, as in the above case. Thus, the avoidance principle holds. In particular, by the results of the preceding section, F_t has a finite collapsing time which is at most equal to that of P_t.

Summarizing, we have the following result.

COROLLARY 5

Let $\overline{M}^{n+1}$ be either a space form $\mathbb{Q}_\epsilon^{n+1}$ or a hyperbolic space $\mathbb{H}_{\mathbb{F}}^m$. Given a Weingarten function $W \in C^\infty(\Gamma)$, assume that $F : M^n \times [0,T) \to \overline{M}^{n+1}$ is a W-flow of a compact Riemannian manifold M such that $F_0(M)$ is contained in an open totally convex ball $B_R \subset \overline{M}^{n+1}$, whose boundary ∂B_R is a strictly convex geodesic sphere of $\overline{M}^{n+1}$. Assume further that one of the following holds:

- *W is odd.*
- *F_t is an embedding with the inward orientation for all $t \in [0,T)$.*

Under these conditions, denoting by

$$P : \mathbb{S}^n \times [0, T_R) \to \overline{M}^{n+1}$$

the parallel W-flow with inward orientation, collapsing time T_R, and initial data $P_0(\mathbb{S}^n) = \partial B_R$, we have that $F_t(M) \cap P_t(\mathbb{S}^n) = \emptyset$ for all $t \in [0,T)$. Consequently, the inequalities $T \le T_R < \infty$ hold.

Proof of Theorem 5

Since F_0 is an embedding, for any sufficiently small $t > 0$, F_t is also an embedding. Let us suppose, by contradiction, that there exists a first time $t_0 > 0$ such that F_{t_0} is not an embedding. In this way,

$$\Omega := \big\{(p,q) \in M \times M;\, p \neq q,\, F_{t_0}(p) = F_{t_0}(q)\big\}$$

is a nonempty compact set of $M \times M$ which is disjoint from the diagonal D of $M \times M$. Thus, there is an open set $U \subset M \times M$ such that $D \subset U$ and Ω is disjoint from the closure of U in $M \times M$.

Now, observing that $V := (M \times M) - U$ is compact in $M \times M$, define the function

$$D(t) = \min_{(p,q)\in V} \operatorname{dist}^2\big(F_t(p), F_t(q)\big), \quad t \in [0, t_0].$$

Since, for a sufficiently small t, F_t is an embedding, for such a t we have $D(t) > 0$. However, proceeding just as in the proof of Theorem 4, we conclude that D is nondecreasing, which contradicts the fact that $D(t_0) = 0$. □

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References

[1] J. Berndt and J. C. Díaz-Ramos, *Homogeneous hypersurfaces in complex hyperbolic spaces*, Geom. Dedicata **138** (2009), 129–150. MR 2469992. DOI 10.1007/s10711-008-9303-8.

[2] J. Berndt, F. Tricerri, and L. Vanhecke, *Generalized Heisenberg Groups and Damek–Ricci Harmonic Spaces*, Lect. Notes Math. **1598**, Springer-Verlag, Berlin 1995. MR 1340192. DOI 10.1007/BFb0076902.

[3] R. L. Bishop and R. J. Crittenden, *Geometry of Manifolds*, Pure Appl. Math. **XV**, Academic Press, New York–London, 1964. MR 0169148.

[4] T. E. Cecil and P. J. Ryan, *Geometry of hypersurfaces*, Springer Monogr. Math., Springer-Verlag, Berlin, 2015. MR 3408101. DOI 10.1007/978-1-4939-3246-7.

[5] I. M. Gel'fand, *Lectures on Linear Algebra*, Interscience Tracts Pure Appl. Math. No. 9, Interscience Publishers, New York–London (1961). MR 0122819.

[6] R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), no. 2, 153–179. MR 0862046.

[7] G. Huisken and A. Polden, "Geometric evolution equations for hypersurfaces" in *Calculus of Variations and Geometric Evolution Problems (Cetraro, 1996)*, Lect. Notes Math. **1713**, Springer, Berlin, 1999, 45–84. MR 1731639. DOI 10.1007/BFb0092669.

[8] S. Kim, Y. Nikolayevsky, and J. Park, *Einstein hypersurfaces of the Cayley projective plane*. Differential Geom. Appl. **69** (2020), 101594, 6. MR 4056219. DOI 10.1016/j.difgeo.2020.101594.

[9] C. Mantegazza, *Lecture Notes on Mean Curvature Flow*, Progr. Math., Birkhäuser/Springer, Basel, 2011. MR 2815949. DOI 10.1007/978-3-0348-0145-4.

[10] H. F. Münzner, *Isoparametrische Hyperflächen in Sphären*, Math. Ann. **251** (1980), no. 1, 57–71. MR 0583825. DOI 10.1007/BF01420281.

[11] H. F. Santos dos Reis and K. Tenenblat, *The mean curvature flow by parallel hypersurfaces*, Proc. Amer. Math. Soc. **146** (2018), no. 11, 4867–4878. MR 3856153. DOI 10.1090/proc/14178.

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