Maximum Principles at Infinity for Surfaces of Bounded Mean Curvature in $\mathbb{R}^3$ and $\mathbb{H}^3$

RONALDO F. DE LIMA & WILLIAM MEEKS, III

ABSTRACT. Let M_1, M_2 be disjoint surfaces in $\mathbb{R}^3$ or $\mathbb{H}^3$ with (possibly empty) boundaries ∂M_1, ∂M_2 and bounded mean curvature. We establish a maximum principle at infinity for these surfaces by proving that under certain conditions on their curvatures, M_1 and M_2 cannot approach each other asymptotically.

1. INTRODUCTION

In the early nineteen fifties, H. Hopf [6] established a general maximum principle for second order linear elliptic partial differential equations. This principle has had several important applications to the global theory of hypersurfaces in n-dimensional Euclidean and hyperbolic spaces. The applications of this principle give rise to the geometric property that two such hypersurfaces with a specified type of contact at a given point, actually coincide in a neighborhood of this point (see Section **2.2** for a precise statement).

Maximum principles at infinity are generalizations of Hopf's maximum principle for surfaces which have a contact at infinity, in the sense that they are disjoint and approach each other asymptotically in a specified way that we call *ideal contact at infinity* (see Definition 3.1 in Section 3).

A general maximum principle at infinity for surfaces in a complete 3-manifold with constant sectional curvature can be stated as follows.

Maximum Principle at Infinity. Suppose M_1, M_2 are two disjoint surfaces with boundaries ∂M_1, ∂M_2, immersed in a 3-manifold $N^3(c)$ with constant sectional curvature c. M_1 and M_2 are said to satisfy the maximum principle at infinity if

$$\operatorname{dist}(M_1, M_2) = \min\{\operatorname{dist}(M_1, \partial M_2), \operatorname{dist}(M_2, \partial M_1)\},$$

where dist stands for the distance function in $N^3(c)$.

Langevin-Rosenberg [8], Meeks-Rosenberg [13] and Soret [15] obtained several versions of the maximum principle at infinity for minimal surfaces in complete

Indiana University Mathematics Journal ©, Vol. 53, No. 5 (2004)

flat 3-manifolds ($c = 0$) and many of these results have had important applications to the global theory of minimal surfaces (cf. [9] and [12]).

In [4], the first author established a maximum principle at infinity for surfaces with nonzero constant mean curvature which are properly embedded in the Euclidean space $\mathbb{R}^3$. For this result, it was assumed that:

(1) The surfaces have bounded Gaussian curvature;
(2) One of the surfaces is parabolic (full harmonic measure on its boundary);
(3) The surfaces have an ideal contact at infinity.

In the present paper, we improve the result in [4] mentioned above, by proving a maximum principle at infinity for surfaces M_1, M_2 with boundary and bounded mean curvature, with M_1 properly embedded in $\mathbb{R}^3$ and M_2 properly immersed in $\mathbb{R}^3$. It is also assumed that M_1 has bounded Gaussian curvature, M_2 has an ideal contact at infinity with M_1, and the mean curvature functions of M_1 and M_2, H_{M_1}, H_{M_2} satisfy $|H_{M_2}| \le c_0 \le H_{M_1}$, $c_0 > 0$ (Theorem 3.2, Section 3).

As an application of Theorem 3.2, we prove that if M_1 is a properly embedded surface in $\mathbb{R}^3$ without boundary, bounded mean curvature $H_{M_1} \ge c_0 > 0$ and bounded Gaussian curvature, then any other properly immersed surface M_2 in $\mathbb{R}^3$ with empty boundary and mean curvature satisfying $|H_{M_2}| \le c_0$, does not lie on the mean convex side of M_1 (Theorem 3.4, Section 3).

Lately, there has been a growing interest in surfaces in hyperbolic space (cf. [3], [7] and the references therein). As seen in the literature, the theory of properly embedded surfaces of (not normalized) mean curvature greater than 2 in hyperbolic space $\mathbb{H}^3$ is akin to the theory of nonzero mean curvature surfaces in $\mathbb{R}^3$. In Section 4 we extend Theorem 3.2, as well as Theorem 3.4, to surfaces of (bounded) mean curvature greater than 2 in $\mathbb{H}^3$. It should be remarked that, as far as we know, there has not been established any maximum principle at infinity for such surfaces in $\mathbb{H}^3$.

2. NOTATION AND PRELIMINARIES

2.1. Notation. Given a smooth oriented surface $M \subset \mathbb{R}^3$, we define on M the *mean curvature function* and the *mean curvature vector* respectively by

$$H_M = k_1 + k_2 \quad \text{and} \quad \mathbf{H}_M = H_M N,$$

where k_i, $i = 1, 2$, are the principal curvatures of M, and N is the unit normal field on M.

We will write dist for the distance function in $\mathbb{R}^3$ and dist_M for the intrinsic distance function on M.

$N_\varepsilon(M)$ will denote the closed ε-disk bundle of the normal bundle of M and, if M has nowhere vanishing mean curvature, $N_\varepsilon^*(M)$ will denote the set of points (p, v) in $N_\varepsilon(M)$ such that $\langle \mathbf{H}(p), v \rangle \ge 0$, where $\langle , \rangle$ is the canonical metric in $\mathbb{R}^3$.

2.2. Hopf's maximum principle for the mean curvature equation. We will recall here Hopf's maximum principle for the mean curvature equation, which is frequently used throughout the paper.

Suppose M_1 and M_2 are two smooth oriented surfaces of $\mathbb{R}^3$ which are tangent at an interior point p and have at p the same oriented normal. In this case, we say that M_1 and M_2 have a *contact* at p, and we say that M_1 *lies above* M_2 *near* p, if when we express M_k, $k = 1, 2$, as graphs of functions f_k over the common tangent plane through p, we have $f_1 \geq f_2$ in a neighborhood of p.

Maximum Principle (Hopf)**.** *Let M_1 and M_2 be oriented surfaces in $\mathbb{R}^3$ which have a contact at a point p. Then, if $H_{M_1} \leq H_{M_2}$ near p, it is not true that M_1 lies above M_2, unless M_1 coincides with M_2 near p.*

2.3. The 1***-parameter family of nodoids.*** We shall briefly recall some properties of the 1-parameter family of immersed Delaunay surfaces known as nodoids. These properties will play a fundamental role in barrier type constructions in the proof of the maximum principle at infinity. For further details and proofs, we refer to Section 3 of [10].

A *nodoid* is a rotational symmetric surface immersed (not embedded) in $\mathbb{R}^3$ whose generating curve is a roulette of a hyperbola.

Given $H > 0$, one proves that there is a 1-parameter family $\{N_t, t > 0\}$ of nodoids of constant mean curvature H. Moreover, for each t, there is a catenoid-shaped piece $\hat{N}_t$ of N_t, which is contained in a slab S_t determined by two parallel planes $P_1(t)$, $P_2(t)$ (Fig. 2.1). Write d_t for the distance between $P_1(t)$ and $P_2(t)$ and let P_0 be the plane parallel to $P_1(t)$ and $P_2(t)$, equidistant to $P_1(t)$ and $P_2(t)$. Then, for each t, one has (cf. Proposition 1 in [10]):

- (i) $\hat{N}_t$ is tangent to $P_1(t)$ and $P_2(t)$;
- (ii) $\hat{N}_t$ is symmetric with respect to $P_0(t)$;
- (iii) $\hat{N}_t$ intersects $P_0(t)$ in a circle of radius t;
- (iv) The mean curvature vector of $\hat{N}_t$ points outward from the bounded domain determined by the slab S_t and $\hat{N}_t$;
- (v) $\hat{N}_t$ lies in the convex hull of its boundary;
- (vi) $\lim_{t\to 0} d_t = 0$.

3. The Euclidean Case

In the proof of the main theorem of this section, we shall use the following result due to A. Ros and H. Rosenberg [14].

We will refer to it as the Ros-Rosenberg Theorem.

Theorem (Ros-Rosenberg)**.** *Let M be a complete stable nonzero constant mean curvature surface in $\mathbb{R}^3$ with nonempty boundary. Then, there exists a constant $C > 0$ such that, for all $p \in M$,*

$$\text{dist}_M(p, \partial M) < C.$$

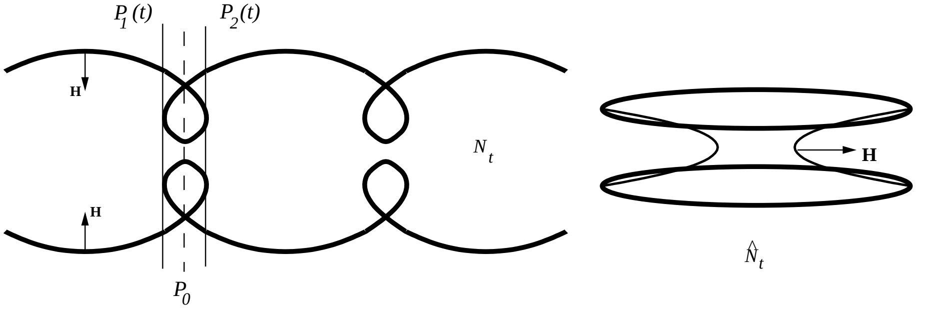

FIGURE 2.1.

Definition 3.1. Let M_1 be a surface which is properly embedded in $\mathbb{R}^3$ and whose mean curvature function is positive. We say that a surface M_2 in $\mathbb{R}^3$ has an *ideal contact at infinity* with M_1, if M_1 and M_2 are disjoint and there are sequences of interior points $p_n \in M_1$, $q_n \in M_2$ and $\lambda_n > 0$ with

$$\operatorname{dist}(p_n, q_n) \to 0 \quad \text{and} \quad q_n - p_n = \lambda_n \mathbf{H}_{M_1}(p_n).$$

Theorem 3.2. *Let M_1 be a surface with boundary ∂M_1 and bounded Gaussian curvature, which is properly embedded in $\mathbb{R}^3$ and whose mean curvature satisfies $c_0 \le H_{M_1} \le c_1$, $c_0, c_1 > 0$. Assume M_2 is a surface with boundary ∂M_2, which is properly immersed in $\mathbb{R}^3$ and such that $|H_{M_2}| \le c_0$. Then, if M_2 has an ideal contact at infinity with M_1, one has*

$$\min\{\operatorname{dist}(M_1, \partial M_2), \operatorname{dist}(M_2, \partial M_1)\} = 0.$$

The ideal contact at infinity extends to asymptotic surfaces the concept of contact at a point given in Section ***2.2***. Example 3.3 below shows that Theorem 3.2 does not hold without this assumption. We would like to thank the Referee for rephrasing our first definition of ideal contact at infinity and making it more appropriate to the context.

Example 3.3. Let M_1 be the cylinder $M_1 = \{(x, y, z) \mid x^2 + y^2 = 1,\ z > z_0\}$ and M_2 the surface of revolution obtained by rotating the curve $\alpha(t) = (t, 0, 1/(t-1))$, $1 < t < t_0$, about the z axis. Since there are no $p \in M_1$, $q \in M_2$ nor $\lambda > 0$ such that $q - p = \lambda \mathbf{H}_{M_1}(p)$, M_2 does not have an ideal contact at infinity with M_1. On the other hand, it is easily seen that M_1 and M_2 are disjoint, $H_{M_1} = 1$, $|H_{M_2}| < 1$, and $0 = \operatorname{dist}(M_1, M_2) < \min\{\operatorname{dist}(M_1, \partial M_2), \operatorname{dist}(M_2, \partial M_1)\}$.

Proof of Theorem 3.2. After possibly rescaling the metric of $\mathbb{R}^3$, we can assume $c_0 = 1$.

Suppose the theorem is false, i.e.,

$$d = \min\{\operatorname{dist}(M_1, \partial M_2), \operatorname{dist}(M_2, \partial M_1)\} > 0.$$

The idea of the proof is to replace M_2 by a stable constant mean curvature 1 surface M, and then derive a contradiction by using the Ros-Rosenberg Theorem. Since the mean curvature and the Gaussian curvature of M_1 are bounded, M_1 has bounded second fundamental form. Thus, for ε small, there are no focal points of M_1 within a distance ε from M_1, which implies that the exponential map from $N_\varepsilon^*(M_1)$ to $\mathbb{R}^3$ is a submersion. Assume that $\varepsilon < d$.

We will work in $N_\varepsilon^*(M_1)$ with the flat metric induced from the exponential map and with M_1 considered to be its zero section. We shall also denote by $M_2 \subset N_\varepsilon^*(M_1)$ the pull back of the surface $M_2 \subset \mathbb{R}^3$ via the exponential map. Notice that $M_2 \subset N_\varepsilon^*(M_1)$ has nonempty interior since $M_2 \subset \mathbb{R}^3$ has an ideal contact at infinity with M_1.

Let $\pi : \widetilde{M}_1 \to M_1$ be the universal covering space of M_1 and endow $\widetilde{M}_1$ with the metric induced by π. Pull back, via π, the disk bundle $N_\varepsilon^*(M_1)$ to obtain the disk bundle $N_\varepsilon^*(\widetilde{M}_1)$, and let $\widetilde{M}_2 \subset N_\varepsilon^*(\widetilde{M}_1)$ be the lifting of $M_2 \subset N_\varepsilon^*(M_1)$ to $N_\varepsilon^*(\widetilde{M}_1)$. It is easily seen that $\operatorname{dist}(\widetilde{M}_1, \widetilde{M}_2) = \operatorname{dist}(M_1, M_2)$ and $\operatorname{dist}(\widetilde{M}_i, \partial\widetilde{M}_j) = \operatorname{dist}(M_i, \partial M_j)$, i, $j = 1, 2$; $i \neq j$. From these considerations, we may assume M_1 is simply connected.

Choose a positive $\delta < \varepsilon/2$ such that δ is less than the intrinsic injectivity radius of M_1 and write $M_1(\delta) = \{p \in M_1 \mid \operatorname{dist}_{M_1}(p, \partial M_1) \geq \delta\}$. Given $p \in M_1 - M_1(\delta)$ satisfying $d_{M_1}(p, \partial M_1) = \delta/2$, let D_p be the intrinsic disk of radius $\delta/4$ on M_1 centered at p, and $D_p^+ \subset N_\varepsilon^*(M_1)$, the δ/k graph over D_p, k a fixed positive integer. For δ small and k large, we have that for any p as above, there exist simple Jordan curves $\sigma_p \subset D_p$, $\sigma_p^+ \subset D_p^+$ bounding an annulus $\mathcal{N}(p)$ in $N_\varepsilon^*(M_1)$, which approximates a piece of a nodoid of constant mean curvature greater than 1 that satisfies properties (i)-(vi) in Section ***2.3***.

Therefore, by construction, $\mathcal{N}(p)$ has mean curvature greater than 1 and mean curvature vector pointing outward from the domain in $N_\varepsilon^*(M_1)$ bounded by the disks D_p, D_p^+ and $\mathcal{N}(p)$ (Fig. 3.1). Moreover, the convex hull property (v) and the choice of small δ imply $\mathcal{N}(p)$ is disjoint from M_2.

We will now proceed to construct the stable constant mean curvature 1 surface M mentioned above, by using parts of M_1, M_2 and $\mathcal{N} = \bigcup_p \mathcal{N}(p)$ as barriers for the Plateau problem for surfaces of constant mean curvature 1.

Let Ω be the closure of a connected component of $N_{\delta/k}^*(M_1) - M_2$ whose boundary intersects the interiors of M_1 and M_2, and denote by $\widehat{M}_2$ the part of M_2 contained in $\partial\Omega$.

Let $\widehat{\Omega}$ be the closure of the component of $\Omega - \mathcal{N}$ that contains $\widehat{M}_2$ and consider in $\widehat{\Omega}$ an exhaustion $\{\widehat{\Omega}_i\}$ by relatively compact domains, that is, $\widehat{\Omega}_1 \subset \widehat{\Omega}_2 \subset \cdots \subset \widehat{\Omega}_i \subset \cdots$ and $\bigcup_i \widehat{\Omega}_i = \widehat{\Omega}$.

Let $A_i \subset M_1$, $B_i \subset \widehat{M}_2$, $C_i \subset \mathcal{N}$ be relatively compact domains such that $A_i = \partial\widehat{\Omega}_i \cap M_1$, $B_i = \partial\widehat{\Omega}_i \cap \widehat{M}_2$, $C_i = \partial\widehat{\Omega}_i \cap \mathcal{N}$ and assume that, for all i, A_i, B_i are nonempty and that $\{B_i\}$ is an exhaustion of $\widehat{M}_2$.

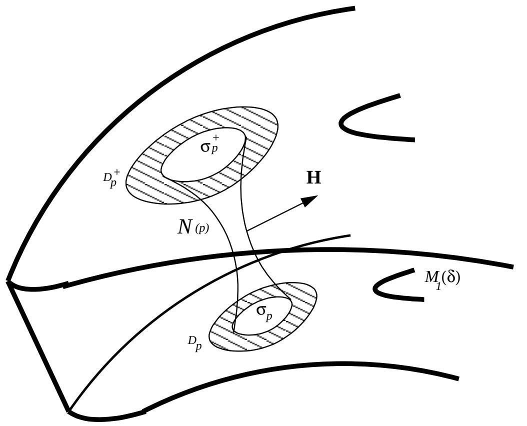

FIGURE 3.1.

In the case $\widehat{M}_2$ has constant mean curvature 1, we will assume that the domains B_i are unstable, otherwise we would set $M = \widehat{M}_2$.

For each i, consider in $\widehat{\Omega}_i$ the Plateau problem for surfaces of constant mean curvature 1 and boundary $\Gamma_i = \partial B_i \subset \widehat{M}_2$. Set $U_i = A_i \cup B_i \cup C_i \subset \partial\widehat{\Omega}_i$ and notice that we can choose $V_i = \partial\widehat{\Omega}_i - U_i$ in such a way that $H_{V_i} > 1$. In the appendix, Section 5, we prove that $\partial\widehat{\Omega}_i$ is a good barrier for this Plateau problem, i.e., for each i, there exists a constant mean curvature 1 surface $S_i \subset \widehat{\Omega}_i$, such that $\partial S_i = \Gamma_i$. Also, since S_i is homologous to B_i rel (Γ_i), if γ is a path in $\widehat{\Omega}_1$ joining a point in A_1 to a point in B_1, separation properties imply that γ has odd intersection number with S_i.

Thus, standard compactness and regularity results yield a subsequence of S_i that converges to a surface S in $\widehat{\Omega}$, disjoint from $\mathcal{N}$ and $M_1(\delta)$, such that away from $\partial\widehat{\Omega}$, S is a properly embedded stable constant mean curvature 1 surface.

Therefore, there is an η, $0 < \eta < \delta/k$, such that the region between M_1 and $M_1 + \eta$ in $N_\varepsilon^*(M_1)$ intersects S in a properly embedded stable constant mean curvature 1 surface M. By Berárd-Hauswirth's curvature estimate for stable constant mean curvature surfaces [2], M has bounded Gaussian curvature away from its boundary. Hence, by considering the part of M in the region between M_1 and $M_1 + \eta'$ in $N_\varepsilon^*(M_1)$, $\eta' < \eta$, we can assume M has bounded Gaussian curvature. Notice also that, by construction, $\operatorname{dist}(M, M_1) < \min\{\operatorname{dist}(M, \partial M_1), \operatorname{dist}(M_1, \partial M)\}$.

After performing a suitable translation of M in $N_\varepsilon^*(M_1)$ (of magnitude $\operatorname{dist}(M, M_1)$), we will have $\operatorname{dist}(M, M_1) = 0$ and $M \cap M_1 = \varnothing$. Otherwise, M and M_1 would have a contact at an interior point $p \in M \cap M_1$ with M above M_1 in the orientation given by $\mathbf{H}_{M_1}$. By Hopf's maximum principle, this contact would imply that M and M_1 coincide along a curve joining p to the boundary of M or M_1, which clearly is a contradiction.

M and M_1 have bounded second fundamental form, so there is an $r > 0$ such that for each $p \in M$, $q \in M_1$, sufficiently away from ∂M, ∂M_1, respectively, M and M_1 are locally graphs of C^1-uniformly bounded functions defined on disks of radius r in T_pM and T_qM_1, centered at the origin. We will refer to these local graphs as r-graphs.

Let $p_n \in M$, $q_n \in M_1$ be sequences of interior points in M and M_1 such that $\text{dist}(p_n, q_n) \to 0$. Notice that since $\min\{\text{dist}(M, \partial M_1), \text{dist}(M_1, \partial M)\} > 0$, for all n, p_n and q_n are away from ∂M and ∂M_1, respectively. Fix a point $O \in N_\varepsilon^*(M_1)$ and, for each n, let φ_n be the translation in $N_\varepsilon^*(M_1)$ that maps p_n to O. Passing to a subsequence, if necessary, we can assume the tangent planes of $\varphi_n(M)$ at O converge to a plane Π containing O. Then, for large n, the r-graphs of $\varphi_n(M)$ at O will be graphs of constant mean curvature 1 over the disk D of radius $r/2$ in Π, centered at O. Due to uniform curvature and area bounds, these graphs will converge to a graph $M(\infty)$ over D, $O \in M(\infty)$, with $H_{M(\infty)} = 1$.

Since $\text{dist}(p_n, q_n) \to 0$, the sequence $\varphi_n(q_n) \to O$. Thus, applying the same reasoning to the local graphs of $\varphi_n(M_1)$ at q_n, we obtain a graph $M_1(\infty)$ over D, $O \in M_1(\infty)$, such that $H_{M_1(\infty)} \geq 1$ and with $M(\infty)$ above $M_1(\infty)$ in the orientation given by $\mathbf{H}_{M_1(\infty)}$. We remark that the convergent plane Π is the same for $M(\infty)$ and $M_1(\infty)$ as M and M_1 are disjoint. Therefore, Hopf's maximum principle gives $M(\infty) = M_1(\infty)$, which implies that the r-graphs of M at p_n approach asymptotically M_1.

Now, for each $p_n \in M$, consider a minimizing geodesic γ_n from p_n to ∂M and notice that, by Ros-Rosenberg Theorem, the length of γ_n is uniformly bounded. Let $\bar{p}_n \in \gamma_n$ be a point on the geodesic γ_n that belongs to the boundary of the r-graph of M at p_n. As before, the r-graphs of M at $\bar{p}_n$ will approach asymptotically M_1. Since the length of γ_n is uniformly bounded, by repeating this reasoning (choosing at each step $\bar{p}_n$ closer to ∂M) we will eventually reach ∂M, which implies $\text{dist}(\partial M, M_1) = 0$ and gives the desired contradiction. ◻

Let M be a surface without boundary, which is properly embedded in $\mathbb{R}^3$. In this case, M separates $\mathbb{R}^3$ into two connected components. If $H_M > 0$, we call *mean convex side* of M, the component to which $\mathbf{H}_M$ points to.

As an application of the maximum principle at infinity established above, we obtain the following result.

Theorem 3.4. *Suppose* M_1 *is a properly embedded surface in* $\mathbb{R}^3$ *without boundary and of bounded Gaussian curvature. Then, if the mean curvature function of* M_1 *satisfies* $c_0 \leq H_{M_1} \leq c_1$, $c_0, c_1 > 0$, *a surface* M_2 *without boundary, which is properly immersed in* $\mathbb{R}^3$ *and whose mean curvature satisfies* $|H_{M_2}| \leq c_0$, *cannot lie on the mean convex side of* M_1.

Proof. Suppose M_2 is on the mean convex side of M_1. If $\text{dist}(M_1, M_2) = 0$, it implies that M_2 has an ideal contact at infinity with M_1. In this case, we can carry out the reasoning of the proof of Theorem 3.2 and, analogously, derive a contradiction.

If $\mathrm{dist}(M_1, M_2) > 0$, there are sequences $p_n \in M_1$ and $q_n \in M_2$, such that the sequence $p_n - q_n$ has a subsequence that converges to a vector $v \in \mathbb{R}^3$ with $\|v\| = \mathrm{dist}(M_1, M_2)$. Replace M_2 by $\overline{M}_2 = M_2 + v$ and notice that $\mathrm{dist}(M_1, \overline{M}_2) = 0$. The previous paragraph implies M_1 and $\overline{M}_2$ cannot be disjoint. Thus, M_1 and $\overline{M}_2$ are tangent at a point $p \in M_1 \cap \overline{M}_2$, with $\overline{M}_2$ above M_1 in the orientation given by $\mathbf{H}_{M_1}$. Thus, Hopf's maximum principle gives $M_1 = \overline{M}_2$. It means M_1 differs from M_2 by a translation in $\mathbb{R}^3$ of magnitude $\mathrm{dist}(M_1, M_2)$ which, in turn, implies that M_1 and M_2 are parallel planes and contradicts $H_{M_1} > 0$. □

Remark. In [14] Ros and Rosenberg have established a maximum principle at infinity for properly embedded surfaces with constant mean curvature in $\mathbb{R}^3$ which is similar to Theorem 3.4 above. It should be remarked that in their result none of the surfaces is assumed to have bounded Gaussian curvature.

4. THE HYPERBOLIC CASE

In the proof of the maximum principle at infinity given in Section 3, we have used some results established for surfaces in Euclidean space $\mathbb{R}^3$, like the Ros-Rosenberg Theorem, Hopf's maximum principle and the curvature estimate of Bérard and Hauswirth [2].

It turns out that all these results have analogous versions for surfaces of mean curvature greater than 2 in hyperbolic space $\mathbb{H}^3$. In particular, the following extension of the Ros-Rosenberg Theorem was obtained in [5].

Theorem 4.1 (de Lima). *Let M be a complete stable surface of constant mean curvature $H > 2$ in $\mathbb{H}^3$ with possibly nonempty boundary. Then, there exists a constant $C > 0$ such that, for all $p \in M$,*

$$\mathrm{dist}_M(p, \partial M) < C.$$

In addition, the construction carried out in the appendix, Section 5, can be easily adapted to surfaces in $\mathbb{H}^3$ (cf. [1]). Therefore, one can mimic the proof of Theorem 3.2 and obtain a maximum principle at infinity for surfaces in $\mathbb{H}^3$ as stated below. The notation is the same as in Section 2 and Section 3, with $\mathbb{R}^3$ replaced by $\mathbb{H}^3$.

Theorem 4.2. *Let M_1 be a surface with boundary ∂M_1 and bounded Gaussian curvature, which is properly embedded in $\mathbb{H}^3$ and whose mean curvature satisfies $2 < c_1 \le H_{M_1} \le c_2$, $c_1, c_2 \in \mathbb{R}$. Assume M_2 is a surface with boundary ∂M_2, which is properly immersed in $\mathbb{H}^3$ and such that $|H_{M_2}| \le c_1$. Then, if M_2 has an ideal contact at infinity with M_1, one has*

$$\min\{\mathrm{dist}(M_1, \partial M_2), \mathrm{dist}(M_2, \partial M_1)\} = 0.$$

As in the Euclidean case, we can apply Theorem 4.2 and obtain the following result.

Theorem 4.3. *Suppose M_1 is a properly embedded surface in $\mathbb{H}^3$ without boundary and of bounded Gaussian curvature. Then, if the mean curvature function of M_1 satisfies $2 < c_1 \leq H_{M_1} \leq c_2$, $c_1, c_2 \in \mathbb{R}$, a surface M_2 without boundary, which is properly immersed in $\mathbb{H}^3$ and whose mean curvature satisfies $|H_{M_2}| \leq c_1$, cannot lie on the mean convex side of M_1.*

Proof. Suppose M_2 is on the mean convex side of M_1. If $\operatorname{dist}(M_1, M_2) = 0$, we can apply the reasoning of the proof of Theorem 4.2 and derive a contradiction.

If $\operatorname{dist}(M_1, M_2) > 0$, as in the Euclidean case, we may replace M_2 by a surface $\overline{M}_2$ with $\operatorname{dist}(M_1, \overline{M}_2) = 0$, obtained by a suitable hyperbolic translation of M_2 of magnitude $\operatorname{dist}(M_1, M_2)$. The first paragraph implies M_1 and $\overline{M}_2$ must have a contact at a point p and, by Hopf's maximum principle, $M_1 = \overline{M}_2$. It means M_1 and M_2 differ in $\mathbb{H}^3$ by a hyperbolic translation of magnitude $\operatorname{dist}(M_1, M_2)$. It is easily seen that it implies M_1 and M_2 are either totally geodesic planes or horospheres of $\mathbb{H}^3$. In both cases it contradicts M_1 has mean curvature greater than 2. ◻

5. Appendix

Let M_1, $M_2 \subset \mathbb{R}^3$ be two complete smooth disjoint surfaces in $\mathbb{R}^3$ and $A \subset M_1$, $B \subset M_2$, relatively compact domains in M_1 and M_2 respectively. Let $\Omega \subset \mathbb{R}^3$ be a relatively compact domain in $\mathbb{R}^3$ such that $\partial\Omega = A \cup B \cup C$, C a surface in $\mathbb{R}^3$, disjoint from $A \cup B$.

Suppose that $H_A \leq -1$, $H_B \leq 1$ and $H_C < -1$, with respect to the *outward* normal of Ω and assume $\Gamma = \partial B \subset M_2$ smooth. We want to prove that there exists a constant mean curvature 1 surface $S \subset \bar{\Omega}$ with $\partial S = \Gamma$, such that S is disjoint from $\partial\Omega$, unless $S = B$. In this case we say that $\partial\Omega$ is a good barrier for this Plateau problem. The existence of such an S is a consequence of the following general fact.

Given a compact oriented surface $M \subset \mathbb{R}^3$, let $\varphi^t : N_\varepsilon(M) \to \mathbb{R}^3$ be the flux

$$\varphi^t(p) = p + tN(\pi(p)), \quad p \in N_\varepsilon(M),\ t \in (-\delta, \delta),$$

where $\pi : N_\varepsilon(M) \to M$ is the normal projection from $N_\varepsilon(M)$ to M, N is the unit normal field of M and ε, δ are assumed to be small enough so that φ^t is well defined.

Given $p \in N_\varepsilon(M)$, let v_1, v_2 be nonparallel vectors in $T_p(N_\varepsilon(M)) \approx \mathbb{R}^3$ and denote by P the parallelogram determined by v_1 and v_2. Let $\mathcal{P}$ be the plane of $\mathbb{R}^3$ through $\varphi^t(p)$ and orthogonal to $N(\pi(p))$. Denote by P_* the parallelogram determined by $\varphi^t_*(p).v_1$, $\varphi^t_*(p).v_2$ and let $P_*^\top$ be the projection of P_* to $\mathcal{P}$. A straightforward calculation gives

$$\operatorname{Area}(P_*^\top) - \operatorname{Area}(P) \leq -tH_M(\pi(p)) \operatorname{Area}(P) + \mathcal{O}(t^2). \tag{5.1}$$

We now prove the existence of the surface S mentioned above.

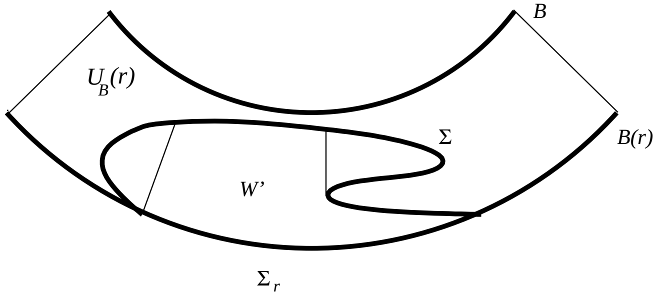

FIGURE 5.1.

Consider a relatively compact domain $B' \subset M_2$ with $B \subset B'$ and a small $\varepsilon > 0$ such that the sets $B(r) = \{p \in \mathbb{R}^3 \mid \operatorname{dist}(p, B') = r\} \cap \Omega$, $0 \le r \le \varepsilon$, define a smooth foliation of $N_\varepsilon(B') \cap \Omega$. Assume also that ε is small enough so that for all $r \in (0, \varepsilon]$, $\Gamma(r) = \partial B(r) \subset \partial\Omega$ is smooth.

Call $U_B(r)$ the connected component of Ω determined by $B(r)$ whose boundary intersects B. Let $W \subset \overline{U_B(r)}$ be a compact domain such that $\partial W = \Sigma \cup \Sigma_r$, Σ, Σ_r smooth, and $\Sigma_r \subset B(r)$. We claim that if r is sufficiently small, then the volume V enclosed by W satisfies

$$V \ge \operatorname{Area}(\Sigma_r) - \operatorname{Area}(\Sigma), \tag{5.2}$$

which is trivial if $\operatorname{Area}(\Sigma_r) \le \operatorname{Area}(\Sigma)$. So, assume that $\operatorname{Area}(\Sigma_r) > \operatorname{Area}(\Sigma)$.

Let f be the distance function from Σ_r to Σ and denote by $\Sigma' \subset \Sigma$ the graph of f in the normal bundle of Σ_r in $\mathbb{R}^3$, i.e., Σ' is the image of Σ_r by the map

$$F: \begin{cases} \Sigma_r \to \mathbb{R}^3, \\ \ p \to p + f(p)\nu(p), \end{cases}$$

where ν is the unit normal field of Σ_r pointing *into* W. It suffices to prove

$$V' \ge \operatorname{Area}(\Sigma_r) - \operatorname{Area}(\Sigma'), \tag{5.3}$$

where V' is the volume of the region $W' \subset W$ determined by Σ_r and Σ' in the normal bundle of Σ_r in $\mathbb{R}^3$ (Fig. 5.1).

Denote respectively by $d\Sigma_r$, $d\Sigma'$ the area element of Σ_r and Σ'. Since we are assuming $H_B \le 1$ and $\operatorname{Area}(\Sigma_r) > \operatorname{Area}(\Sigma)$, it follows from (5.1) that for all $p \in \Sigma_r$ and $v_1, v_2 \in T_p\Sigma_r$ one has

$$d\Sigma_r(p)(v_1, v_2) - d\Sigma'(F(p))(F_* v_1, F_* v_2) < f(p)\, d\Sigma_r(p)(v_1, v_2) + \mathcal{O}(r^2),$$

which, for small r, implies (5.3) and proves the claim.

Analogously, define the sets $A(r)$, $U_A(r)$ and consider a domain $\widetilde{W} \subset \overline{U_A(r)}$ such that $\partial\widetilde{W} = \widetilde{\Sigma} \cup \widetilde{\Sigma}_r$, $\widetilde{\Sigma}$, $\widetilde{\Sigma}_r$ smooth and $\widetilde{\Sigma}_r \subset A(r)$. Then, since $H_A \leq -1$, a reasoning similar to the one in the previous paragraphs gives

$$\text{(5.4)} \qquad \operatorname{Area}(\widetilde{\Sigma}) - \operatorname{Area}(\widetilde{\Sigma}_r) \geq \widetilde{V},$$

where $\widetilde{V}$ is the volume enclosed by $\widetilde{W}$.

Now, for each r set $U(r) = U_A(r) \cup U_B(r)$, $\Omega(r) = \Omega - U(r)$.

Consider compact domains $Q \subset \Omega$ such that $\partial Q = \Sigma \cup B(r)$, where Σ is an oriented surface with $\partial\Sigma = \partial B(r) = \Gamma(r)$. Consider in $B(r)$ the orientation given by the *outward* normal of $\Omega(r)$ and let J denote the functional on $(Q, \partial Q)$,

$$\text{(5.5)} \qquad (Q, \partial Q) \to \operatorname{Area}(\Sigma) + \operatorname{Vol}(Q),$$

where $\operatorname{Vol}(Q)$ denotes the *algebraic* volume of Q in the given orientation of $B(r)$, i.e., Vol is positive inside $U_B(r)$ and negative outside $U_B(r)$. It is well known that J attains a minimum in a region $Q(r)$ with $\partial Q(r) = S(r) \cup B(r)$ and the smooth points of $S(r)$ have constant mean curvature 1 (see [1, 11]).

Let Q_i be a sequence of regions in Ω with $\partial Q_i = \Sigma_i \cup B(r)$ and $Q_i \to Q(r)$. From the claim above, we can assume that Q_i does not intersect $U(r)$. Indeed, if Q_i intersected $U(r)$, we could replace Q_i by a region $\widetilde{Q}_i \subset \Omega(r)$ by cutting off the pieces of Q_i inside $U(r)$ (Fig. 5.2). Then, inequalities (5.2) and (5.4) would give $J(\widetilde{Q}_i) \leq J(Q_i)$, which implies $J(\widetilde{Q}_i) \to J(Q(r))$ as $i \to \infty$.

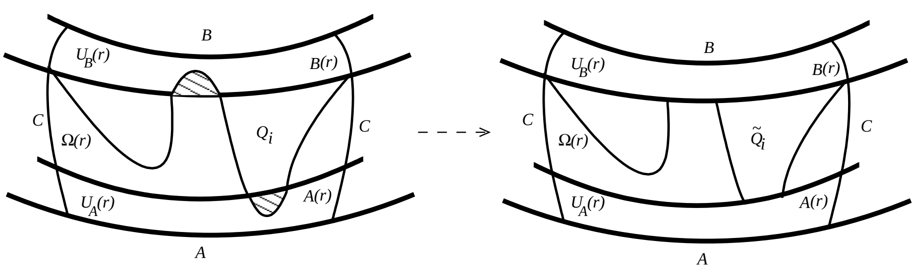

FIGURE 5.2.

Therefore, $S(r) \subset \Omega(r)$. Furthermore, since $S(r)$ minimizes J in Ω, this together with regularity results ([1, 11]) imply that $S(r)$ is smooth away from $\partial\Omega$. Now, C is known to be a good barrier for $H_C < -1$, which implies that $S(r)$ and C are disjoint and $S(r)$ is smooth.

Since $\Gamma(r) \to \Gamma$ and $B(r) \to B$ as $r \to 0$, by standard compactness results, there is a subsequence of $S(r)$ that converges to a smooth constant mean curvature 1 surface S in $\bar{\Omega}$ with $\partial S = \Gamma$.

Observe that by continuity, the domain $Q' \subset \bar{\Omega}$ bounded by S and B is a minimum for the functional J given by (5.5), defined on domains $Q \subset \Omega$ with $\partial Q = \Sigma \cup B$, Σ an oriented surface with $\partial\Sigma = \partial B = \Gamma$.

It remains to prove that S does not intersect $\partial\Omega$, unless $S = B$.

Since A and B are disjoint, $H_A \le -1$ and $H_C < -1$, by Hopf's maximum principle S is disjoint from $A \cup C$.

Now, suppose S and B have a contact at an interior point $p \in S \cap B$ and notice that, in this case, the mean curvature vector of S at p points outward from Ω. Otherwise, since variations in the direction of the mean curvature vector decrease area and Vol is negative inside Ω, we could strictly reduce J by varying in Ω a neighborhood U of p, $U \subset S$. Since $H_B \le 1 = H_S$, Hopf's maximum principle implies $S = B$; as desired.

Acknowledgments. First author was supported by CNPq, and second author was supported by NSF grant DMS - 0104044.

The first author would like to thank the GANG laboratory at the University of Massachusetts at Amherst for their generous hospitality during the preparation of this paper.

REFERENCES

[1] HILARIO ALENCAR and HAROLD ROSENBERG, *Some remarks on the existence of hypersurfaces of constant mean curvature with a given boundary, or asymptotic boundary, in hyperbolic space*, Bull. Sci. Math. **121** (1997), 61–69. MR 97j:53007

[2] PIERRE BÉRARD and LAURENT HAUSWIRTH, *General curvature estimates for stable H-surfaces immersed into a space form*, Bull. Sci. Math. (9) **78** (1999), 667–700. MR 2000h:53080 (English, with English and French summaries)

[3] PASCAL COLLIN, LAURENT HAUSWIRTH, and Harold ROSENBERG, *The geometry of finite topology Bryant surfaces*, Ann. of Math. (2) **153** (2001), 623–659. MR 2002j:53012

[4] RONALDO F. DE LIMA, *A maximum principle at infinity for surfaces with constant mean curvature in Euclidean space*, Ann. Global Anal. Geom. **20** (2001), 325–343, http://dx.doi.org/10.1023/A:1013050905779. MR 2003d:53014

[5] ______, *On surfaces with constant mean curvature in hyperbolic space*, Illinois J. Math. **47** (2003), 1079–1098. 2 036 991

[6] HEINZ HOPF, Differential Geometry in the Large, Lecture Notes in Mathematics, vol. 1000, Springer-Verlag, Berlin, 1989, ISBN 3-540-51497-X. MR 90f:53001

[7] NICHOLAS J. KOREVAAR, ROB KUSNER, WILLIAM H. MEEKS III, and BRUCE SOLOMON, *Constant mean curvature surfaces in hyperbolic space*, Amer. J. Math. **114** (1992), 1–43. MR 92k:53116

[8] RÉMI LANGEVIN and HAROLD ROSENBERG, *A maximum principle at infinity for minimal surfaces and applications*, Duke Math. J. **57** (1988), 819–828. MR 90c:53025

[9] HIPPOLYTE LAZARD-HOLLY and WILLIAM H. MEEKS III, *Classification of doubly-periodic minimal surfaces of genus zero*, Invent. Math. **143** (2001), 1–27. MR 2001m:53013

[10] RAFAEL LÓPEZ, *Constant mean curvature graphs in a strip of* $\mathbf{R}^2$, Pacific J. Math. **206** (2002), 359–373. MR 2003j:53010

[11] UMBERTO MASSARI, *Esistenza e regolarità delle ipersuperfice di curvatura media assegnata in* $\mathbf{R}^n$, Arch. Rational Mech. Anal. **55** (1974), 357–382, http://dx.doi.org/10.1007/BF00250439. MR 50 #8240 (Italian)

[12] WILLIAM H. MEEKS III, Joaquín PÉREZ, and Antonio ROS, *Uniqueness of the Riemann minimal examples*, Invent. Math. **133** (1998), 107–132, http://dx.doi.org/10.1007/s002220050241. MR 99b:53013

[13] WILLIAM H. MEEKS III and HAROLD ROSENBERG, *The maximum principle at infinity for minimal surfaces in flat three manifolds*, Comment. Math. Helv. **65** (1990), 255–270. MR 91d:53011

[14] ANTONIO ROS and HAROLD ROSENBERG, *Properly embedded surfaces with constant mean curvature* (preprint).

[15] MARC SORET, *Maximum principle at infinity for complete minimal surfaces in flat* 3*-manifolds*, Ann. Global Anal. Geom. **13** (1995), 101–116. MR 96j:53007

RONALDO F. DE LIMA:
Departamento de Matemática
Centro de Ciências Exatas e da Terra
Universidade Federal do Rio Grande do Norte
Lagoa Nova, CEP 59.072-970, Brasil
E-MAIL: ronaldo@ccet.ufrn.br.

WILLIAM MEEKS, III:
Department of Mathematics and Statistics
University of Massachusetts at Amherst
Lederle Graduate Research Tower
Amherst, MA 01003-9305, U. S. A. .
E-MAIL: bill@gang.umass.edu

KEY WORDS AND PHRASES: constant mean curvature, maximum principle
2000 MATHEMATICS SUBJECT CLASSIFICATION: 53A10 (53A35)
Received: *September 21st, 2003; revised: April 25th, 2004.*