

# TRANSLATING SOLITONS TO FLOWS BY POWERS OF THE GAUSSIAN CURVATURE IN RIEMANNIAN PRODUCTS

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ABSTRACT. We consider translating solitons to flows by positive powers  $\alpha$  of the Gaussian curvature — called  $K^\alpha$ -flows — in Riemannian products  $M \times \mathbb{R}$ . We prove that, when  $M$  is the Euclidean space  $\mathbb{R}^n$ , the sphere  $S^n$ , or one of the hyperbolic spaces  $\mathbb{H}_p^n$ , there exist complete rotational translating solitons to  $K^\alpha$ -flow in  $M \times \mathbb{R}$  for certain values of  $\alpha$ .

## 1. INTRODUCTION

Over the last decades, the subject of extrinsic curvature flows in Riemannian manifolds has experienced a significant development. Along this time, special attention has been given to mean curvature and Gaussian curvature flows in Euclidean space, resulting in achievements such as the proof of short time existence of solutions, and of their convergence (after rescaling) to round spheres. (For a thorough account of these and other extrinsic flows, we refer the recent book [3].)

Likewise, flows by powers of the Gaussian curvature, which we call  $K^\alpha$ -flows, have been considered by many authors, most notably B. Andrews (see, e.g., [2, 4, 5, 7, 9, 16, 20]). Among other reasons, the interest in  $K^\alpha$ -flows comes from the fact that they naturally relate to a wide range of research fields, varying from image processing to affine geometry and geometric analysis (cf. [1]).

In the context of extrinsic curvature flows in Euclidean space, there are special hypersurfaces, called *translating solitons*, which have the distinguished property of moving by translation under such a flow. In fact, the construction and classification of complete translating solitons constitutes a major problem in this theory (see, e.g., [10, 18, 19, 20]). On this regard, we point out that J. Urbas [20] established, for all  $\alpha \in (0, 1/2]$ , the existence of complete rotational translating solitons to the  $K^\alpha$ -flow in Euclidean space.

More generally, translating solitons can be defined in Riemannian products  $M \times \mathbb{R}$ , where  $M$  is an arbitrary Riemannian manifold. In [17], assuming  $M$  complete with non-positive sectional curvature and rotationally invariant metric, Lira and Martín obtained a one-parameter family of rotational translating solitons to the mean curvature flow in  $M \times \mathbb{R}$ . Also, in [14, 15], results of the same nature were obtained in the context of Lorentzian products. On the other hand, to our knowledge, there are no works on translating solitons to  $K^\alpha$ -flows in products  $M \times \mathbb{R}$ .

In this paper, we prove existence of complete rotational translating solitons to the  $K^\alpha$ -flow in  $M \times \mathbb{R}$  (for certain positive values of  $\alpha$ ) when  $M$  is the Euclidean

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space  $\mathbb{R}^n$ , the sphere  $\mathbb{S}^n$ , or one of the hyperbolic spaces  $\mathbb{H}_{\mathbb{F}}^m$  (rank-one symmetric spaces of noncompact type). In particular, we recover the aforementioned result of Urbas, giving it a new proof.

We remark that the translating solitons we construct here are all bowl-type graphs of radial functions. In the Euclidean and hyperbolic cases, these graphs are entire, whereas in the spherical case they project onto an open hemisphere  $B$  of  $\mathbb{S}^n$ , being asymptotic to the half-cylinder  $\partial B \times [0, +\infty)$ .

The paper is organized as follows. In Section 2, we fix notation regarding hypersurfaces of Riemannian products  $M \times \mathbb{R}$ . In Section 3, we discuss graphs defined on families of parallel hypersurfaces, setting some formulae. In Section 4, we introduce the  $K^\alpha$ -flow in  $M \times \mathbb{R}$  and establish a key lemma. Finally, in Section 5, we state and prove our main result.

## 2. PRELIMINARIES

Given a Riemannian manifold  $M^n$ ,  $n \geq 2$ , consider the Riemannian product  $M \times \mathbb{R}$  endowed with its standard metric

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_M + dt^2,$$

and denote its Levi-Civita connection by  $\bar{\nabla}$ .

Let  $\Sigma^n$  be an oriented hypersurface of  $M \times \mathbb{R}$ . Set  $N$  for its unit normal field and  $A$  for its shape operator with respect to  $N$ , so that

$$AX = -\bar{\nabla}_X N, \quad X \in T\Sigma,$$

where  $T\Sigma$  stands for the tangent bundle of  $\Sigma$ . The principal curvatures of  $\Sigma$ , that is, the eigenvalues of the shape operator  $A$ , will be denoted by  $k_1, \dots, k_n$ . In this setting, the *Gaussian curvature* of  $\Sigma$  is the function  $K : \Sigma \rightarrow \mathbb{R}$  defined as

$$K := \det A = k_1 \dots k_n.$$

We denote by  $\partial_t$  the gradient of the projection  $\pi_{\mathbb{R}}$  of  $M \times \mathbb{R}$  on the factor  $\mathbb{R}$ . Given a hypersurface  $\Sigma \subset M \times \mathbb{R}$ , its *height function*  $\xi$  and its *angle function*  $\Theta$  are defined by the following identities:

$$\xi(x) := \pi_{\mathbb{R}}|_{\Sigma} \quad \text{and} \quad \Theta(x) := \langle N(x), \partial_t \rangle, \quad x \in \Sigma.$$

We shall denote the gradient of  $\xi$  on  $\Sigma$  by  $T$ , so that

$$(1) \quad T = \partial_t - \Theta N.$$

Given  $t \in \mathbb{R}$ , the set  $P_t := M \times \{t\}$  is called a *horizontal hyperplane* of  $M \times \mathbb{R}$ . Horizontal hyperplanes are all isometric to  $M$  and totally geodesic in  $M \times \mathbb{R}$ . In this context, we call a transversal intersection  $\Sigma_t := \Sigma \bar{\cap} P_t$  a *horizontal section* of  $\Sigma$ . Any horizontal section  $\Sigma_t$  is a hypersurface of  $P_t$ . So, at any point  $x \in \Sigma_t \subset \Sigma$ , the tangent space  $T_x \Sigma$  of  $\Sigma$  at  $x$  splits as the orthogonal sum

$$(2) \quad T_x \Sigma = T_x \Sigma_t \oplus \text{Span}\{T\}.$$

**Definition 1.** A hypersurface  $\Sigma \subset M \times \mathbb{R}$  is called *rotational*, if there exists a fixed point  $o \in M$  such that any connected component of any horizontal section  $\Sigma_t$  of  $\Sigma$  is contained in a geodesic sphere of  $M \times \{t\}$  with center at  $o \times \{t\}$ . If so, the set  $\{o\} \times \mathbb{R}$  is called the *axis* of  $\Sigma$ .

Recall that the rank-one symmetric spaces of non-compact type are the hyperbolic spaces  $\mathbb{H}_{\mathbb{R}}^m$ ,  $\mathbb{H}_{\mathbb{C}}^m$ ,  $\mathbb{H}_{\mathbb{K}}^m$  and  $\mathbb{H}_{\mathbb{O}}^2$ ,  $m \geq 1$ , called *real hyperbolic space*, *complex hyperbolic space*, *quaternionic hyperbolic space* and *Cayley hyperbolic plane*, respectively. We will adopt the unified notation  $\mathbb{H}_{\mathbb{F}}^m$  for these spaces with  $m = 2$  for  $\mathbb{F} = \mathbb{O}$ . The real dimension of  $\mathbb{H}_{\mathbb{F}}^m$  is  $n = m \dim \mathbb{F}$ .

We remark that the hyperbolic spaces  $\mathbb{H}_{\mathbb{F}}^m$  are all Einstein–Hadamard manifolds whose sectional curvatures are negative and pinched. In particular, their geodesic spheres are all strictly convex (see, e.g., [6]).

### 3. GRAPHS ON PARALLEL HYPERSURFACES

Consider an isometric immersion

$$f : M_0^{n-1} \rightarrow M^n$$

between two Riemannian manifolds  $M_0^{n-1}$  and  $M^n$ , and suppose that there is a neighborhood  $\mathcal{U}$  of  $M_0$  in  $TM_0^\perp$  without focal points of  $f$ , that is, the restriction of the normal exponential map  $\exp_{M_0}^\perp : TM_0^\perp \rightarrow M$  to  $\mathcal{U}$  is a diffeomorphism onto its image. In this case, denoting by  $\eta$  the unit normal field of  $f$ , there is an open interval  $I \ni 0$  such that, for all  $p \in M_0$ , the curve

$$(3) \quad \gamma_p(s) = \exp_M(f(p), s\eta(p)), \quad s \in I,$$

is a well defined geodesic of  $M$  without conjugate points. Thus, for all  $s \in I$ ,

$$f_s : \begin{array}{ccc} M_0 & \rightarrow & M \\ p & \mapsto & \gamma_p(s) \end{array}$$

is an immersion of  $M_0$  into  $M$ , which is said to be *parallel* to  $f$ . Observe that, given  $p \in M_0$ , the tangent space  $f_{s*}(T_p M_0)$  of  $f_s$  at  $p$  is the parallel transport of  $f_*(T_p M_0)$  along  $\gamma_p$  from 0 to  $s$ . We also remark that, with the induced metric, the unit normal  $\eta_s$  of  $f_s$  at  $p$  is given by

$$\eta_s(p) = \gamma_p'(s).$$

**Definition 2.** Let  $\phi : I \rightarrow \phi(I) \subset \mathbb{R}$  be an increasing diffeomorphism, i.e.,  $\phi' > 0$ . With the above notation, we call the set

$$(4) \quad \Sigma := \{(f_s(p), \phi(s)) \in M \times \mathbb{R}; p \in M_0, s \in I\},$$

the *graph* determined by  $\{f_s; s \in I\}$  and  $\phi$ , or  $(f_s, \phi)$ -*graph*, for short.

For an arbitrary point  $x = (f_s(p), \phi(s))$  of an  $(f_s, \phi)$ -graph  $\Sigma$ , one has

$$T_x \Sigma = f_{s*}(T_p M_0) \oplus \text{Span}\{\partial_s\}, \quad \partial_s = \eta_s + \phi'(s)\partial_t.$$

So, a unit normal to  $\Sigma$  is

$$(5) \quad N = \frac{-\phi'}{\sqrt{1+(\phi')^2}}\eta_s + \frac{1}{\sqrt{1+(\phi')^2}}\partial_t.$$

In particular, its angle function is

$$(6) \quad \Theta = \frac{1}{\sqrt{1+(\phi')^2}}.$$

As shown in [11, Theorem 6]), any  $(f_s, \phi)$ -graph  $\Sigma$  has the *T-property*, meaning that  $T$  is a principal direction at any point of  $\Sigma$ . More precisely,

$$(7) \quad AT = \frac{\phi''}{(\sqrt{1+(\phi')^2})^3}T.$$

Conversely, any hypersurface of  $M \times \mathbb{R}$  with non vanishing angle function having  $T$  as a principal direction is given locally as an  $(f_s, \phi)$ -graph.

Given an  $(f_s, \phi)$ -graph  $\Sigma$ , let  $\{X_1, \dots, X_n\}$  be the orthonormal frame of principal directions of  $\Sigma$  in which  $X_n = T/\|T\|$ . In this case, for  $1 \leq i \leq n-1$ , the fields  $X_i$  are all horizontal, that is, tangent to  $M$  (cf. (2)). Therefore, setting

$$(8) \quad \varrho := \frac{\phi'}{\sqrt{1 + (\phi')^2}}$$

and considering (5), we have, for all  $i = 1, \dots, n-1$ , that

$$k_i = \langle AX_i, X_i \rangle = -\langle \bar{\nabla}_{X_i} N, X_i \rangle = \varrho \langle \bar{\nabla}_{X_i} \eta_s, X_i \rangle = -\varrho k_i^s,$$

where  $k_i^s$  is the  $i$ -th principal curvature of  $f_s$ . Also, it follows from (7) that  $k_n = \varrho'$ . Thus, the principal curvatures of the  $(f_s, \phi)$ -graph  $\Sigma$  at  $(f_s(p), \phi(s)) \in \Sigma$  are

$$(9) \quad k_i = -\varrho(s)k_i^s(p) \quad (1 \leq i \leq n-1) \quad \text{and} \quad k_n = \varrho'(s).$$

In particular, the Gaussian curvature  $K$  of  $\Sigma$  at  $(f_s(p), \phi(s))$  is given by

$$(10) \quad K = (-\varrho(s))^{n-1} K_s(p) \varrho'(s),$$

where  $K_s$  denotes the Gaussian curvature of the hypersurface  $f_s$ .

We remark that the function  $\varrho$  defined in (8) determines the function  $\phi$ . Indeed, it follows from equality (8) that

$$(11) \quad \phi(s) = \int_{s_0}^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du + \phi(s_0), \quad s_0, s \in I.$$

In addition, from (6) and (8), the unit normal  $N$  defined in (5) can be written as  $N = -\varrho \eta_s + \Theta \partial_t$ . Hence, the relation

$$(12) \quad \Theta = \sqrt{1 - \varrho^2}$$

holds everywhere on any  $(f_s, \phi)$ -graph  $\Sigma$ . In particular, one has  $\varrho = \|T\|$ .

Now, let us assume that the family of parallel hypersurfaces

$$\mathcal{F} := \{f_s : M_0 \rightarrow M; s \in I\}$$

is *isoparametric*, that is, for each  $s \in I$ , any principal curvature  $k_i^s$  of  $f_s \in \mathcal{F}$  is constant (possibly depending on  $i$  and  $s$ ). We shall assume further that each  $f_s$  is strictly convex with  $k_i^s < 0$ .

In this setting, the Gaussian curvature  $K_s$  of each hypersurface  $f_s$  is a (non vanishing) function of  $s$  alone. Hence, from (10), the same is true for any  $(f_s, \phi)$ -graph  $\Sigma$  built on  $\mathcal{F}$ , that is,

$$(13) \quad K(f_s(p), \phi(s)) = (-\varrho(s))^{n-1} \varrho'(s) K_s, \quad (f_s(p), \phi(s)) \in \Sigma.$$

It should also be noticed that, by equalities (9), such a graph  $\Sigma$  is strictly convex.

#### 4. TRANSLATING $K^\alpha$ -SOLITONS IN $M \times \mathbb{R}$

Given a real number  $\alpha \neq 0$ , we say that an oriented strictly convex hypersurface  $\Sigma$  of  $M \times \mathbb{R}$  *moves under  $K^\alpha$ -flow* if there exists a one-parameter family of immersions  $F: M_0 \times [0, u_0) \rightarrow M \times \mathbb{R}$ ,  $u_0 \leq +\infty$ , such that

$$(14) \quad \begin{cases} \frac{\partial F}{\partial u}^\perp(p, u) = K^\alpha(p, u)N(p, u). \\ F(M_0, 0) = \Sigma, \end{cases}$$

where  $N(p, u)$  is the inward unit normal to the hypersurface  $F_u := F(\cdot, u)$ ,  $K(p, u)$  is the Gaussian curvature of  $F_u$  with respect to  $N_u := N(\cdot, u)$ , and  $\frac{\partial F}{\partial u}^\perp$  denotes the normal component of  $\frac{\partial F}{\partial u}$ , i.e.,

$$\frac{\partial F}{\partial u}^\perp = \left\langle \frac{\partial F}{\partial u}, N_u \right\rangle N_u.$$

In particular, the first equality in (14) is equivalent to

$$(15) \quad \left\langle \frac{\partial F}{\partial u}(p, u), N(p, u) \right\rangle = K^\alpha(p, u).$$

We call such a map  $F$  a  $K^\alpha$ -flow in  $M \times \mathbb{R}$ .

Denote by  $\exp$  the exponential map of  $M \times \mathbb{R}$  and consider an isometric immersion  $F_0: M_0 \rightarrow M \times \mathbb{R}$ . Define then the map

$$F(p, u) := \exp_{F_0(p)}(u\partial_t), \quad (p, u) \in M_0 \times [0, +\infty),$$

and notice that, for each  $u \in (0, +\infty)$ , the hypersurface  $F(M_0, u)$  is nothing but a vertical translation of  $\Sigma := F(M_0, 0)$ . Since vertical translations are isometries of  $M \times \mathbb{R}$  keeping the second factor invariant, we have that  $\Sigma$  and  $F(M_0, u)$  are congruent with coinciding angle functions and Gaussian curvature, that is,

$$(16) \quad \Theta(p, u) = \Theta(p, 0) \quad \text{and} \quad K(p, u) = K(p, 0) \quad \forall (p, u) \in M_0 \times [0, u_0).$$

Now, differentiating  $F$  with respect to  $u$ , we have

$$(17) \quad \frac{\partial F}{\partial u}(p, u) = (d\exp_{F_0(p)})(u\partial_t)\partial_t = \partial_t.$$

From (16) and (17), we have that  $F$  satisfies (15) if and only if the equality

$$\Theta(p, 0) = K^\alpha(p, 0)$$

holds on  $M_0$ . This fact motivates the following concept.

**Definition 3.** Given  $\alpha > 0$ , we say that an oriented strictly convex hypersurface  $\Sigma$  of the Riemannian product  $M \times \mathbb{R}$  is a *translating soliton to the  $K^\alpha$ -curvature flow* (or simply a *translating  $K^\alpha$ -soliton*) if the equality

$$K^\alpha = \Theta$$

holds everywhere on  $\Sigma$ . From the above discussion, any such hypersurface  $\Sigma$  is the initial data of a  $K^\alpha$ -flow by vertical translations.

Let us consider now an  $(f_s, \phi)$ -graph  $\Sigma$  in  $M \times \mathbb{R}$  such that the family

$$\mathcal{F} := \{f_s : M_0 \rightarrow M; s \in I\}$$

is isoparametric with  $k_i^s < 0$ . In this case, it follows from (12) and (13) that  $\Sigma$  is a translating  $K^\alpha$ -soliton if and only if its associated  $\varrho$  function satisfies

$$(18) \quad (-\varrho(s))^{n-1} \varrho'(s) K_s = (1 - \varrho^2(s))^{1/2\alpha}, \quad s \in I.$$

Since  $K_s$  never vanishes on  $I$ , we have that (18) holds for  $\varrho$  if and only if

$$(19) \quad \tau'(s) = \frac{(-1)^{n-1} n (1 - \tau^{2/n})^{1/2\alpha}}{K_s}, \quad \tau := \varrho^n.$$

Summarizing, we have the following result.

**Lemma 1.** *Let  $\Sigma$  be an  $(f_s, \phi)$ -graph in  $M \times \mathbb{R}$  whose associated family*

$$\mathcal{F} := \{f_s : M_0 \rightarrow M ; s \in I\}$$

*of oriented parallel hypersurfaces is isoparametric with each of them having negative principal curvatures. Then,  $\Sigma$  is a translating  $K^\alpha$ -soliton if and only if (19) holds for  $\tau := \varrho^n$ , where  $\varrho : I \rightarrow \mathbb{R}$  is as in (8).*

## 5. THE MAIN RESULT

For  $n \geq 2$  and  $m \geq 1$ , let  $M$  be one of the following manifolds, each of them endowed with its canonical Riemannian metric: Euclidean space  $\mathbb{R}^n$ , the sphere  $\mathbb{S}^n$ , or one of the hyperbolic spaces  $\mathbb{H}_{\mathbb{F}}^m$ . Then, define

$$(20) \quad \mathcal{R}_M := \begin{cases} +\infty & \text{if } M \neq \mathbb{S}^n \\ \pi/2 & \text{if } M = \mathbb{S}^n \end{cases}$$

and consider a family

$$(21) \quad \mathcal{F} := \{f_s : \mathbb{S}^{n-1} \rightarrow M ; s \in (0, \mathcal{R}_M)\}$$

of concentric geodesic spheres of  $M$  indexed by their radiuses, that is, for a fixed point  $o \in M$ , and for each  $s \in (0, \mathcal{R}_M)$ ,  $f_s(\mathbb{S}^{n-1})$  is the (strictly convex) geodesic sphere  $S_s(o)$  of  $M$  with center at  $o$  and radius  $s$ . In accordance to the notation of Section 3, for each  $s \in (0, \mathcal{R}_M)$ , we choose the outward orientation of  $f_s$ , so that any principal curvature  $k_i^s$  of  $f_s$  is negative.

For  $M = \mathbb{H}_{\mathbb{F}}^m$ , the principal curvatures  $k_i^s$  of the geodesic spheres  $f_s \in \mathcal{F}$  are:

$$(22) \quad \begin{aligned} k_1^s &= -\frac{1}{2} \coth(s/2) \text{ with multiplicity } n - p - 1. \\ k_2^s &= -\coth(s) \text{ with multiplicity } p, \end{aligned}$$

where  $n = \dim \mathbb{H}_{\mathbb{F}}^m$ ,  $p = n - 1$  for  $\mathbb{H}^n$ ,  $p = 1$  for  $\mathbb{H}_{\mathbb{C}}^m$ ,  $p = 3$  for  $\mathbb{H}_{\mathbb{K}}^m$ , and  $p = 7$  for  $\mathbb{H}_{\mathbb{O}}^2$  (see, e.g., [8, pgs. 353, 543] and [13]). Thus, the Gaussian curvature  $K_s$  of the geodesic sphere  $S_s(o)$  of  $\mathbb{H}_{\mathbb{F}}^m$  is given by

$$K_s = (-1)^{n-1} \left( \frac{1}{2} \coth(s/2) \right)^{n-p-1} (\coth s)^p.$$

In particular, the function

$$\frac{(-1)^{n-1}}{K_s} = (2 \tanh(s/2))^{n-p-1} (\tanh s)^p$$

is well defined and nonnegative in  $[0, +\infty)$ .

As is well known, in the cases  $M = \mathbb{R}^n$  and  $M = \mathbb{S}^n$  one has

$$\frac{(-1)^{n-1}}{K_s} = s^{n-1} \quad \text{and} \quad \frac{(-1)^{n-1}}{K_s} = (\tan s)^{n-1},$$

respectively.

It is easily seen that, in any of these three cases, the equality

$$(23) \quad \lim_{s \rightarrow \mathcal{R}_M} \int_0^s \frac{(-1)^{n-1}}{K_v} dv = +\infty.$$

holds. Finally, set

$$\begin{aligned} \bullet \delta_n &:= \begin{cases} 1/2 & \text{if } n = 2. \\ \max\left\{\frac{1}{4}, \frac{1}{n-1}\right\} & \text{if } n > 2. \end{cases} \\ \bullet \mathcal{I}_M &:= \begin{cases} (0, 1/2] & \text{if } M \neq \mathbb{S}^n. \\ [\delta_n, 1/2] & \text{if } M = \mathbb{S}^n. \end{cases} \end{aligned}$$

With this notation, we now state and prove our main result.

**Theorem 1.** *Let  $M$  and  $\mathcal{I}_M \subset (0, 1/2]$  be as above. Then, for all  $\alpha \in \mathcal{I}_M$ , there exists a complete rotational strictly convex translating  $K^\alpha$ -soliton in the closed half-space  $M \times [0, +\infty)$ , whose height function is unbounded. In addition, the following assertions hold:*

- For  $M \neq \mathbb{S}^n$ ,  $\Sigma$  is an entire graph over  $M$ .
- For  $M = \mathbb{S}^n$ ,  $\Sigma$  is a graph over an open hemisphere  $B \subset \mathbb{S}^n$ , being asymptotic to the half-cylinder  $\partial B \times [0, +\infty)$ .

*Proof.* Let  $\mathcal{F}$  be a family of parallel geodesic spheres of  $M$  as in (21). We intend to determine a function  $\phi \in C^\infty[0, \mathcal{R}_M)$  such that the corresponding  $(f_s, \phi)$ -graph  $\Sigma$  becomes the desired  $K^\alpha$ -soliton. With this purpose, let us consider the equality (19) as an ODE with variable  $\tau$ , and rewrite it as

$$(24) \quad \frac{d\tau}{(1 - \tau^{2/n})^{1/2\alpha}} = \frac{(-1)^{n-1}n}{K_s} ds.$$

In order to solve (24), consider the functions

$$\Phi: [0, 1) \rightarrow [0, +\infty) \quad \text{and} \quad \Psi: [0, \mathcal{R}_M) \rightarrow [0, +\infty)$$

given by

$$\Phi(\tau) := \int_0^\tau \frac{du}{(1 - u^{2/n})^{1/2\alpha}} \quad \text{and} \quad \Psi(s) := \int_0^s \frac{(-1)^{n-1}n}{K_v} dv.$$

It follows from (23) that  $\Psi$  is a diffeomorphism. Let us show that the same is true for  $\Phi$ . Indeed,  $\Phi$  is a  $C^\infty$  function with positive derivative, which implies that it is a diffeomorphism over its image. So, it remains to prove that  $\Phi([0, 1)) = [0, +\infty)$ . To this end, notice first that  $1 - u^{2/n} \leq 1 - u \forall u \in [0, 1)$ . Therefore, setting  $p := 1/2\alpha \geq 1$ , one has

$$\Phi(\tau) \geq \int_0^\tau \frac{du}{(1 - u)^p} = \begin{cases} \log\left(\frac{1}{1-\tau}\right) & \text{if } p = 1. \\ \frac{(1-\tau)^{1-p} - 1}{p-1} & \text{if } p > 1, \end{cases}$$

which implies that  $\Phi(\tau) \rightarrow +\infty$  as  $\tau \rightarrow 1$ . Hence,  $\Phi([0, 1)) = [0, +\infty)$ , as asserted.

Now, we can define  $\tau: [0, \mathcal{R}_M) \rightarrow \mathbb{R}$  by

$$(25) \quad \tau(s) = \Phi^{-1}(\Psi(s)),$$

which is clearly a solution of (24) (and so of (19)) satisfying

$$(26) \quad 0 = \tau(0) \leq \tau(s) < 1 \quad \forall s \in [0, \mathcal{R}_M) \quad \text{and} \quad \lim_{s \rightarrow \mathcal{R}_M} \tau(s) = 1.$$

Therefore, by Lemma 1, the corresponding  $(f_s, \phi)$ -graph  $\Sigma$  is a translating  $K^\alpha$ -soliton in  $M \times \mathbb{R}$ . Moreover, since  $\varrho = \tau^{1/n}$ , we have from (11) (with  $s_0 = 0$ ) and

(26) that the function  $\phi$  is defined in  $[0, \mathcal{R}_M)$  and satisfies

$$\phi(0) = \phi'(0) = 0 \quad \text{and} \quad \lim_{s \rightarrow \mathcal{R}_M} \phi'(s) = +\infty.$$

We conclude from the above discussion that, if  $M \neq \mathbb{S}^n$ , then  $\Sigma$  is an unbounded entire graph over  $M$  which is contained in the closed half-space  $M \times [0, +\infty)$ . In particular,  $\Sigma$  is complete (Fig. 1).

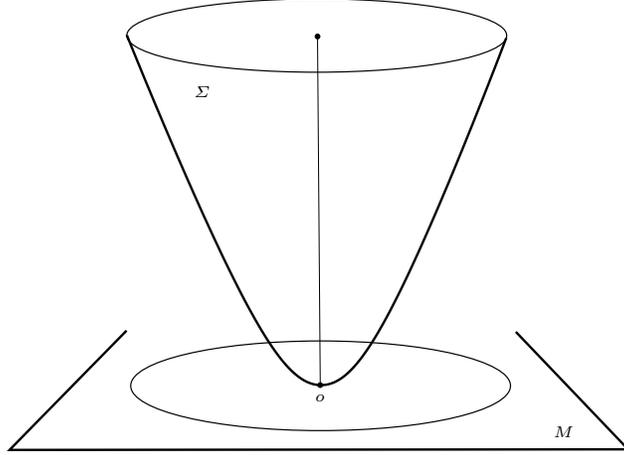


FIGURE 1. A translating  $K^\alpha$ -soliton in  $M \times \mathbb{R}$ .

If  $M = \mathbb{S}^n$ , then  $\Sigma$  is a graph over the open hemisphere  $B$  centered at  $o \in \mathbb{S}^n$  (Fig. 2). Thus, to conclude that  $\Sigma$  is complete and asymptotic to  $\partial B \times [0, +\infty)$ , we must prove that  $\phi$  is unbounded. With this purpose, let us show first that  $\varrho'$  is bounded in  $(0, \pi/2)$ .

For  $n = 2$ , we have from the hypothesis that  $\alpha = 1/2$ . In this case, it is easily checked that  $\varrho(s) = \sin s$  is the solution of (18) satisfying  $\varrho(0) = 0$ . In particular,  $\varrho'(s) = \cos s$  is bounded in  $(0, \pi/2)$ .

For  $n > 2$ , we have from equality (19) that

$$\tau'(s) = n(\tan s)^{n-1}(1 - (\tau(s))^{2/n})^p, \quad p = 1/2\alpha.$$

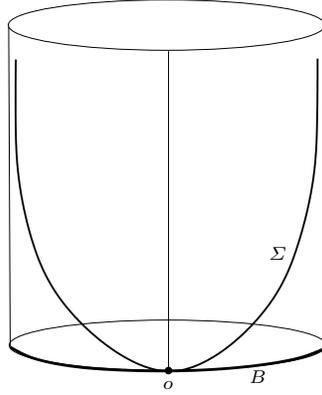
A straightforward calculation gives that, for  $s > 0$ ,

$$(27) \quad \tau''(s) = \mu_1(s) \left[ (n-1) - 2np(\tau(s))^{\frac{2-n}{n}} (\sin s)^2 \frac{\tau'(s)}{\mu_2(s)} \right],$$

where

$$\mu_1(s) := n(\tan s)^{n-2}(\sec s)^2(1 - (\tau(s))^{2/n})^p \quad \text{and} \quad \mu_2(s) := (1 - (\tau(s))^{2/n}) \tan s.$$

Let us suppose, by contradiction, that there exists a sequence  $(s_k)$  in  $(0, \pi/2)$  such that  $\tau'(s_k) \rightarrow +\infty$ . We can assume, without loss of generality, that  $\tau''(s_k) > 0$  for all  $k \in \mathbb{N}$ . Under this assumption, we have from (27) that the sequence  $\mu_2(s_k)$  is unbounded above. Otherwise, for a sufficiently large  $k$ , the expression in the brackets would be negative for  $s = s_k$ , which would give  $\tau''(s_k) < 0$ .


 FIGURE 2. A translating  $K^\alpha$ -soliton in  $\mathbb{S}^n \times \mathbb{R}$ .

So, we can assume  $\mu_2(s_k) \rightarrow +\infty$ . However, from our choice of  $\alpha$ , we have that  $0 \leq p-1 \leq 1$ . Therefore,

$$\frac{\tau'(s_k)}{\mu_2(s_k)} = n(\tan s_k)^{n-2}(1 - (\tau(s_k))^{2/n})^{p-1} \geq n(\tan s_k)^{n-3}\mu_2(s_k) \rightarrow +\infty,$$

which, together with equality (27), gives  $\tau''(s_k) < 0$  for a sufficiently large  $k$  — again a contradiction. Therefore,  $\varrho'$  is bounded in  $(0, \pi/2)$ .

Now, let  $C > 0$  be such that  $\varrho'(s) < C$  for all  $s \in (0, \pi/2)$ . Since we are assuming  $\alpha \geq n-1$ , equality (18) yields

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}} du = \int_0^s \frac{(\tan u)^{\alpha(n-1)}}{(\varrho(u))^{\alpha(n-1)-1}(\varrho'(u))^\alpha} du \geq \frac{1}{C^\alpha} \int_0^s \tan u du,$$

which implies that  $\phi(s) \rightarrow +\infty$  as  $s \rightarrow \pi/2$ . This concludes the proof.  $\square$

We finish by noting that, for a given Riemannian manifold  $M$ , we have from Lemma 1 that there exist local translating  $K^\alpha$ -solitons in  $M \times \mathbb{R}$  as long as  $M$  admits families of isoparametric, strictly convex hypersurfaces. This applies, for instance, to the hyperbolic spaces  $\mathbb{H}_{\mathbb{R}}^n$  and their families of parallel horospheres (see Proposition-(vi), pg 88, in [6]), to the real hyperbolic space  $\mathbb{H}_{\mathbb{R}}^n$  and its families of equidistant hypersurfaces to a given totally geodesic hyperplane, as well as to  $\mathbb{S}^n$  and any of its many families of strictly convex isoparametric hypersurfaces (cf. [12] and the references therein).

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