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# Totally umbilical hypersurfaces of product spaces

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**Abstract.** Given a Riemannian manifold  $M$ , and an open interval  $I \subset \mathbb{R}$ , we characterize nontrivial totally umbilical hypersurfaces of the product  $M \times I$ —as well as of warped products  $I \times_{\omega} M$ —as those which are local graphs built on isoparametric families of totally umbilical hypersurfaces of  $M$ . By means of this characterization, we fully extend to  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$  the results by Souam and Toubiana on the classification of totally umbilical surfaces in  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$ . It is also shown that an analogous classification holds for arbitrary warped products  $I \times_{\omega} \mathbb{S}^n$  and  $I \times_{\omega} \mathbb{H}^n$ .

## 1. Introduction

In submanifold theory, one of the primary concerns is the determination of the totally umbilical hypersurfaces of a given ambient space  $\bar{M}$ . Here, we consider this problem when  $\bar{M}$  is a product manifold  $M \times I$ , and more generally, a warped product  $I \times_{\omega} M$ , where  $I \subset \mathbb{R}$  is an open interval, and  $M$  is a Riemannian manifold.

Trivial examples of totally umbilical hypersurfaces of  $M \times I$ , which are in fact totally geodesic, are the horizontal sections  $M \times \{t\}$ ,  $t \in I$ , and the vertical cylinders over totally geodesic hypersurfaces of  $M$  (if any). Our first main result establishes that nontrivial totally umbilical (resp. totally geodesic) hypersurfaces exist in  $M \times I$  if and only if  $M$  admits local isoparametric families of totally umbilical (resp. totally geodesic) hypersurfaces. (Recall that a family of hypersurfaces is called *isoparametric* if its members are parallel and have constant mean curvature.)

More precisely, we show that a nontrivial totally umbilical (resp. totally geodesic) hypersurface of  $M \times \mathbb{R}$  is necessarily a local graph over an open set of  $M$ , whose level sets constitute an isoparametric family of totally umbilical (resp. totally geodesic) hypersurfaces. Conversely, any such graph is totally umbilical (resp. totally geodesic) in  $M \times \mathbb{R}$ . This result extends naturally to warped products  $I \times_{\omega} M$ , since such a manifold is conformal to  $M \times F(I)$ , where  $F' = 1/\omega$ , and umbilicity is preserved under conformal diffeomorphisms. As a first consequence

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of these results, we recover the characterization of totally umbilical hypersurfaces of  $M \times I$  and  $I \times_{\omega} M$  given by Souam and Van der Veken in [9].

In [8], Souam and Toubiana classified the totally umbilical surfaces in  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$ . They constructed nontrivial complete totally umbilical hypersurfaces in these spaces, all invariant by rotational or translational isometries, and proved that they are properly embedded, analytic, and unique up to ambient isometries.

By means of our characterization of totally umbilical hypersurfaces of product spaces, we not only provide new proofs of Souam and Toubiana's results, but also fully extend them to  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$ . In addition, we show that an analogous classification of the totally umbilical hypersurfaces of  $I \times_{\omega} \mathbb{S}^n$  and  $I \times_{\omega} \mathbb{H}^n$  holds for any warping function  $\omega$ .

It should be mentioned that, in [7], a complete classification of totally umbilical submanifolds of arbitrary codimension in  $\mathbb{S}^n \times \mathbb{R}$  was obtained. When the codimension is one, this classification coincides with ours. Nevertheless, the methods we employ here are different from the ones applied in that work, being our proofs somewhat simpler.

The paper is organized as follows. In Sect. 2, we introduce some notation and formulae. In Sect. 3, we discuss on graphs of  $M \times \mathbb{R}$  (called  $(f_s, \phi)$ -graphs) whose level hypersurfaces are parallel in  $M$ . In Sect. 4, we establish three key lemmas, and in Sect. 5 we characterize totally umbilical hypersurfaces of  $M \times \mathbb{R}$  and  $I \times_{\omega} M$  by means of  $(f_s, \phi)$ -graphs. Finally, in Sects. 6 and 7, we classify the totally umbilical hypersurfaces of  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$ , as well as those of  $I \times_{\omega} \mathbb{S}^n$  and  $I \times_{\omega} \mathbb{H}^n$ .

## 2. Preliminaries

Let  $\Sigma^n$ ,  $n \geq 2$ , be an oriented hypersurface of a Riemannian manifold  $\overline{M}^{n+1}$ . Set  $\overline{\nabla}$  for the Levi-Civita connection of  $\overline{M}$ ,  $N$  for the unit normal field of  $\Sigma$ , and  $A$  for its shape operator with respect to  $N$ , so that

$$AX = -\overline{\nabla}_X N, \quad X \in T\Sigma,$$

where  $T\Sigma$  stand for the tangent bundle of  $\Sigma$ . The principal curvatures of  $\Sigma$ , that is, the eigenvalues of the shape operator  $A$ , will be denoted by  $k_1, \dots, k_n$ .

**Definition 1.** A hypersurface  $\Sigma$  of  $\overline{M}$  is called *totally umbilical* if its shape operator writes as  $A = \lambda \text{Id}$ , where  $\lambda$  is a differentiable function on  $\Sigma$  (called its *umbilical function*) and  $\text{Id}$  is the identity operator on  $T\Sigma$ . A totally umbilical hypersurface with null umbilical function is called *totally geodesic*.

The ambient spaces  $\overline{M}$  we shall consider are the Riemannian products  $M \times I$  and the warped products  $I \times_{\omega} M$ , whose metrics are

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_M + dt^2 \quad \text{and} \quad \langle \cdot, \cdot \rangle = dt^2 + \omega^2 \langle \cdot, \cdot \rangle_M,$$

respectively. Here,  $M$  is an arbitrary Riemannian manifold,  $I \subset \mathbb{R}$  is an open interval, and  $\omega$  a differentiable positive function on  $I$ .

Let  $\overline{M}$  be either of these product spaces. We denote by  $\partial_t$  the gradient of the projection  $\pi_I$  of  $\overline{M}$  on the factor  $I$ . Notice that, in contrast with the Riemannian product case, in the warped product case,  $\partial_t$  is not a parallel field.

Given a hypersurface  $\Sigma$  of  $\overline{M}$ , its *height function*  $\xi$  and its *angle function*  $\Theta$  are defined by the following identities:

$$\xi(x) := \pi_I|_{\Sigma} \quad \text{and} \quad \Theta(x) := \langle N(x), \partial_t \rangle, \quad x \in \Sigma.$$

We shall denote the gradient of  $\xi$  on  $\Sigma$  by  $T$ , so that

$$T = \partial_t - \Theta N. \quad (1)$$

**Definition 2.** A point of  $\Sigma$  at which  $T$  vanishes will be called *horizontal*.

Given  $t \in \mathbb{R}$ , the set  $P_t := M \times \{t\}$  is called a *horizontal hyperplane* of  $M \times \mathbb{R}$ . Horizontal hyperplanes are all isometric to  $M$  and totally geodesic in  $M \times \mathbb{R}$ . In this context, we call a transversal intersection  $\Sigma_t := \Sigma \bar{\cap} P_t$  a *horizontal section* of  $\Sigma$ . Any horizontal section  $\Sigma_t$  is a hypersurface of  $P_t$ . So, at any point  $x \in \Sigma_t \subset \Sigma$ , the tangent space  $T_x \Sigma$  of  $\Sigma$  at  $x$  splits as the orthogonal sum

$$T_x \Sigma = T_x \Sigma_t \oplus \text{Span}\{T\}. \quad (2)$$

Analogously, we define a *vertical hyperplane*  $P_t = \{t\} \times_{\omega} M$  in  $I \times_{\omega} M$ , and a *vertical section*  $\Sigma_t := \Sigma \bar{\cap} P_t$  of a hypersurface  $\Sigma$  of  $I \times_{\omega} M$ . We point out that the vertical hyperplanes  $P_t$  are totally umbilical in  $I \times_{\omega} M$  with constant umbilical function  $\omega'(t)/\omega(t)$  (see, e.g., [1]).

Given a totally geodesic hypersurface  $\Sigma_0$  of  $M$ , it is easily checked that the products  $\Sigma_0 \times I$  and  $I \times_{\omega} \Sigma_0$  are totally geodesic in  $M \times I$  and  $I \times_{\omega} M$ , respectively. These hypersurfaces, together with the horizontal hyperplanes of  $M \times \mathbb{R}$  and vertical hyperplanes of  $I \times_{\omega} M$ , will be referred as the *trivial umbilical hypersurfaces* of these product spaces.

We will adopt the unified notation  $\mathbb{Q}_{\epsilon}^n$  for the simply connected space form of constant sectional curvature  $\epsilon \in \{1, -1\}$ , so that  $\mathbb{Q}_1^n$  is the unit sphere  $\mathbb{S}^n$ , and  $\mathbb{Q}_{-1}^n$  is the hyperbolic space  $\mathbb{H}^n$ .

Finally, we recall that a hypersurface  $\Sigma \subset \mathbb{Q}_{\epsilon}^n \times I$  is called *rotational* if it is invariant by a one-parameter group of rotations of  $\mathbb{Q}_{\epsilon}^n \times I$  fixing an *axis*  $\{o\} \times I$ ,  $o \in \mathbb{Q}_{\epsilon}^n$ . Notice that, such a  $\Sigma$  is foliated by (parts of) horizontal  $(n-1)$ -spheres with center on the axis, so that they project on a family of parallel geodesic spheres of  $M$  centered at  $o$ .

### 3. Graphs on parallel hypersurfaces

In this section, we introduce graphs in  $M \times I$  which are built on families of parallel hypersurfaces of  $M$ . Through them, we shall construct and classify totally umbilical hypersurfaces of  $M \times I$ .

With this purpose, consider an isometric immersion

$$f : M_0^{n-1} \rightarrow M^n$$

between two Riemannian manifolds  $M_0^{n-1}$  and  $M^n$ , and suppose that there is a neighborhood  $\mathcal{U}$  of  $M_0$  in  $TM_0^\perp$  without focal points of  $f$ , that is, the restriction of the normal exponential map  $\exp_{M_0}^\perp : TM_0^\perp \rightarrow M$  to  $\mathcal{U}$  is a diffeomorphism onto its image. In this case, denoting by  $\eta$  the unit normal field of  $f$ , there is an open interval  $I_0 \ni 0$ , such that, for all  $p \in M_0$ , the curve

$$\gamma_p(s) = \exp_M(f(p), s\eta(p)), \quad s \in I_0, \quad (3)$$

is a well defined geodesic of  $M$  without conjugate points. Thus, for all  $s \in I_0$ ,

$$\begin{aligned} f_s : M_0 &\rightarrow M \\ p &\mapsto \gamma_p(s) \end{aligned}$$

is an immersion of  $M_0$  into  $M$ , which is said to be *parallel* to  $f$ . Observe that, given  $p \in M_0$ , the tangent space  $f_{s*}(T_p M_0)$  of  $f_s$  at  $p$  is the parallel transport of  $f_*(T_p M_0)$  along  $\gamma_p$  from 0 to  $s$ . We also remark that, with the induced metric, the unit normal  $\eta_s$  of  $f_s$  at  $p$  is given by

$$\eta_s(p) = \gamma_p'(s).$$

**Definition 3.** Let  $\phi : I_0 \rightarrow \phi(I_0) \subset \mathbb{R}$  be an increasing diffeomorphism, i.e.,  $\phi' > 0$ . With the above notation, we call the set

$$\Sigma := \{(f_s(p), \phi(s)) \in M \times \mathbb{R}; p \in M_0, s \in I_0\}, \quad (4)$$

the *graph* determined by  $\{f_s; s \in I_0\}$  and  $\phi$ , or  $(f_s, \phi)$ -*graph*, for short.

Notice that, for a given  $(f_s, \phi)$ -graph  $\Sigma$ , and for any  $s \in I_0$ ,  $f_s(M_0)$  is the projection on  $M$  of the horizontal section  $\Sigma_{\phi(s)} \subset \Sigma$ , that is, these sets are the level hypersurfaces of  $\Sigma$ .

For an arbitrary point  $x = (f_s(p), \phi(s))$  of such a graph  $\Sigma$ , one has

$$T_x \Sigma = f_{s*}(T_p M_0) \oplus \text{Span} \{\partial_s\}, \quad \partial_s = \eta_s + \phi'(s)\partial_t.$$

So, a unit normal to  $\Sigma$  is

$$N = \frac{-\phi'}{\sqrt{1 + (\phi')^2}} \eta_s + \frac{1}{\sqrt{1 + (\phi')^2}} \partial_t. \quad (5)$$

In particular, its angle function is

$$\Theta = \frac{1}{\sqrt{1 + (\phi')^2}}. \quad (6)$$

As shown in [5, Theorem 6]), any  $(f_s, \phi)$ -graph  $\Sigma$  has the  $T$ -property, meaning that  $T$  is a principal direction at any point of  $\Sigma$ . More precisely,

$$AT = \frac{\phi''}{(\sqrt{1 + (\phi')^2})^3} T. \quad (7)$$

Conversely, any hypersurface of  $M \times I$  with non vanishing angle function having  $T$  as a principal direction is given locally as an  $(f_s, \phi)$ -graph.

Given an  $(f_s, \phi)$ -graph  $\Sigma$ , let  $\{X_1, \dots, X_n\}$  be the orthonormal frame of principal directions of  $\Sigma$  in which  $X_n = T/\|T\|$ . In this case, for  $1 \leq i \leq n - 1$ , the fields  $X_i$  are all horizontal, that is, tangent to  $M$  (cf. (2)). Therefore, setting

$$\varrho := \frac{\phi'}{\sqrt{1 + (\phi')^2}} \quad (8)$$

and considering (5), we have, for all  $i = 1, \dots, n - 1$ , that

$$k_i = \langle AX_i, X_i \rangle = -\langle \bar{\nabla}_{X_i} N, X_i \rangle = \varrho \langle \bar{\nabla}_{X_i} \eta_s, X_i \rangle = -\varrho k_i^s,$$

where  $k_i^s$  is the  $i$ th principal curvature of  $f_s$ . Also, it follows from (7) that  $k_n = \varrho'$ . Thus, the array of principal curvatures of the  $(f_s, \phi)$ -graph  $\Sigma$  is

$$k_i = -\varrho k_i^s \quad (1 \leq i \leq n - 1) \quad \text{and} \quad k_n = \varrho'. \quad (9)$$

The function defined in (8)—to be called the  $\varrho$ -function of the  $(f_s, \phi)$ -graph  $\Sigma$ —will play a major role in the sequel. We remark that, up to a constant, the  $\varrho$ -function of  $\Sigma$  determines its  $\phi$ -function. Indeed, it follows from equality (8) that

$$\phi(s) = \int_{s_0}^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du + \phi(s_0), \quad s_0, s \in I_0. \quad (10)$$

We conclude this section by introducing a special type of family of parallel hypersurfaces which will be of special interest to us.

**Definition 4.** We call a family  $\{f_s : M_0 \rightarrow M ; s \in I_0\}$  of parallel hypersurfaces *isoparametric* if, for each  $s \in I_0$ ,  $f_s$  has constant mean curvature (depending on  $s$ ). If so, each hypersurface  $f_s$  is also called *isoparametric*.

#### 4. Three key lemmas

The proofs of our main theorems will rely on the following three lemmas. The first establishes conditions on the  $\varrho$ -function of an  $(f_s, \phi)$ -graph under which the resulting hypersurface is totally umbilical. The second states that any nontrivial totally umbilical hypersurface of  $M \times I$  is represented locally as such a graph. In the third, we prove the convergence of a certain improper integral which will be recurrent in the paper.

**Lemma 1.** Let  $\{f_s : M_0 \rightarrow M; s \in I_0\}$  be an isoparametric family of totally umbilical hypersurfaces of  $M$ , and denote by  $\lambda = \lambda(s)$  the umbilical constant of  $f_s$ . Let  $\varrho$  be a solution of the differential equation

$$y' + \lambda(s)y = 0, \quad s \in I_0. \quad (11)$$

Then, if  $0 < \varrho < 1$ , the  $(f_s, \phi)$ -graph  $\Sigma$  determined by  $\varrho$  is a totally umbilical hypersurface of the product  $M \times \mathbb{R}$ .

*Proof.* Let  $\varrho$  be a solution of (11) satisfying  $0 < \varrho < 1$ . From (9), we have that the first  $n - 1$  principal curvatures of the  $(f_s, \phi)$ -graph  $\Sigma$  determined by  $\varrho$  are all equal to  $-\lambda(s)\varrho(s)$ . However,  $k_n = \varrho' = -\lambda\varrho$ , which implies that  $\Sigma$  is totally umbilical in  $M \times \mathbb{R}$ .  $\square$

Regarding Eq. (11), recall that its general solution is

$$\varrho(s) = c \exp\left(-\int_{s_0}^s \lambda(u)du\right), \quad s_0, s \in I_0, \quad c \in \mathbb{R}. \quad (12)$$

**Lemma 2.** Let  $\Sigma \subset M \times I$  be a totally umbilical hypersurface with no horizontal points and non vanishing angle function. Then,  $\Sigma$  is given locally by an  $(f_s, \phi)$ -graph, where  $\{f_s : M_0 \rightarrow M; s \in I_0\}$  is an isoparametric family of totally umbilical hypersurfaces of  $M$ . Moreover, the following assertions are equivalent:

- (i)  $\Sigma$  is totally geodesic.
- (ii) The angle function of  $\Sigma$  is locally constant.
- (iii)  $f_s$  is totally geodesic for all  $s \in I_0$ .

*Proof.* Since  $\Sigma$  is umbilical with no horizontal points, it has the  $T$ -property. So, by [5, Theorem 6],  $\Sigma$  is locally an  $(f_s, \phi)$ -graph. Denoting by  $k(x)$  the principal curvature of  $\Sigma$  at  $x = (f_s(p), \phi(s))$ , we have from equalities (9) that

$$\varrho'(s) = k(x) = -\varrho(s)k_i^s(p) \quad \forall i = 1, \dots, n - 1.$$

Thus, for all  $p \in M_0$  and  $i = 1, \dots, n - 1$ , the following equality holds:

$$k_i^s(p) = -\frac{\varrho'(s)}{\varrho(s)},$$

which implies that  $f_s$  is totally umbilical and isoparametric.

Regarding the assertions (i)–(iii), we have just to notice that, since  $k_n = \varrho'$ ,  $\Sigma$  is totally geodesic if and only if  $\varrho$  is constant. From this, the last equality above, and the identities (6) and (8), one easily concludes that (i), (ii) and (iii) are equivalent.  $\square$

**Lemma 3.** Let  $\varrho : [a, b] \rightarrow \mathbb{R}$  be a differentiable function such that  $0 \leq \varrho \leq 1$ . Assume that either of the following hold:

- (i)  $\varrho(a) \neq 1$ ,  $\varrho(b) = 1$ , and  $\varrho'(b) > 0$ .
- (ii)  $\varrho(a) = 1$ ,  $\varrho(b) \neq 1$ , and  $\varrho'(a) < 0$ .

Under any of these conditions, the improper integral

$$\int_a^b \frac{\varrho(s)}{\sqrt{1-\varrho^2(s)}} ds$$

is convergent.

*Proof.* Assume that (i) occurs. In this case, there exist positive constants,  $\delta$  and  $C$ , such that  $\varrho'(s) \geq C > 0 \forall s \in (b-\delta, b)$ . Therefore,

$$\begin{aligned} \int_{b-\delta}^b \frac{\varrho(s) ds}{\sqrt{1-\varrho^2(s)}} &\leq \int_{b-\delta}^b \frac{\varrho'(s) ds}{\varrho'(s) \sqrt{1-\varrho^2(s)}} \leq \frac{1}{C} \int_{\varrho(b-\delta)}^1 \frac{d\varrho}{\sqrt{1-\varrho^2}} \\ &= \frac{1}{C} \left( \frac{\pi}{2} - \arcsin(\varrho(b-\delta)) \right) \leq \frac{\pi}{2C}, \end{aligned}$$

which proves the case (i). The proof for the case (ii) is analogous.

## 5. Totally umbilical hypersurfaces of $M \times I$ and $I \times_{\omega} M$

In this section, we give a complete characterization of totally umbilical hypersurfaces of the product spaces  $M \times I$  and  $I \times_{\omega} M$ . To begin with, let us observe that Lemmas 1 and 2 immediately give the following result.

**Theorem 1.** *There exist nontrivial totally umbilical (resp. totally geodesic) hypersurfaces in  $M \times I$  if and only if  $M$  admits isoparametric families of totally umbilical (resp. totally geodesic) hypersurfaces. Furthermore, if  $\Sigma \subset M \times \mathbb{R}$  is totally umbilical and connected, then  $\Sigma$  is totally geodesic if and only if its angle function is constant.*

It is a well known fact that  $\mathbb{Q}_{\epsilon}^n$  does not have isoparametric families of totally geodesic hypersurfaces. Hence, we have

**Corollary 1.** *A totally geodesic hypersurface of  $\mathbb{Q}_{\epsilon}^n \times \mathbb{R}$  is necessarily trivial.*

If we let  $M$  be a warped product  $I_0 \times_{\omega} M_0^{n-1}$ , then the standard isometric immersions  $f_s : M_0 \hookrightarrow \{s\} \times_{\omega} M_0 \subset M$ ,  $s \in I_0$ , define a family  $\mathcal{F} = \{f_s ; s \in I_0\}$  of parallel and totally umbilical hypersurfaces of  $M$  whose constant umbilical function is  $\omega'/\omega$  (see Sect. 2). In particular,  $\mathcal{F}$  is isoparametric. Hence, from Theorem 1, there exist non trivial totally umbilical hypersurfaces in  $M \times \mathbb{R}$ .

Conversely, assume that  $\mathcal{F} = \{f_s : M_0 \rightarrow M' \subset M ; s \in I_0\}$  is an isoparametric family of totally umbilical hypersurfaces covering an open set  $M' \subset M$ . In this case, denoting by  $\lambda(s)$  the umbilical constant of  $f_s$ , we have that  $M'$  is isometric to the warped product  $I_0 \times_{\omega} M_0$ , where the warping function  $\omega$  satisfies  $\lambda = \omega'/\omega$  (see, e.g., [2, Section 4]).

From these considerations and Theorem 1, we obtain the following result, which was previously obtained in [9].

**Corollary 2.** *There exists a nontrivial totally umbilical hypersurface  $\Sigma$  in  $M \times I$  if and only if  $M$  contains an open set which is isometric to a warped product  $I_0 \times_{\omega} M_0$  as above. If so,  $\Sigma$  is totally geodesic if and only if the corresponding warping function is constant.*

Given a warped product  $I \times_{\omega} M$ , consider the diffeomorphism

$$F : I \rightarrow J = F(I) \quad (13)$$

such that  $F' = 1/\omega$ . As can be easily checked, the map

$$\begin{aligned} \varphi : I \times_{\omega} M &\rightarrow M \times J \\ (t, p) &\mapsto (p, F(t)) \end{aligned} \quad (14)$$

is a conformal diffeomorphism whose conformal factor is  $F' = 1/\omega$ .

It is a well known fact that the property of being totally umbilical is preserved under a conformal diffeomorphism between the ambient spaces. Therefore, a hypersurface  $\Sigma \subset I \times_{\omega} M$  is totally umbilical if and only if  $\varphi(\Sigma)$  is totally umbilical in  $M \times J$ . This fact, together with Theorem 1, gives the following characterization of umbilical hypersurfaces of  $I \times_{\omega} M$ .

**Theorem 2.** *There exist nontrivial totally umbilical hypersurfaces in  $I \times_{\omega} M$  if and only if  $M$  admits isoparametric families of totally umbilical hypersurfaces.*

In the above theorem, as in the Riemannian product case, the condition on the existence of isoparametric families of totally umbilical hypersurfaces can be replaced by that of  $M$  having a local structure of a warped product.

## 6. Totally umbilical hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

Our aim in this section is to extend the main results by Souam and Toubiana [8] on totally umbilical hypersurfaces of  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$  to the spaces  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$ . Actually, our approach includes the case  $n = 2$  and provides new proofs for Souam and Toubiana's theorems.

### 6.1. Totally umbilical hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$

In the next two theorems, we show that there exist unique (up to ambient isometries) properly embedded nontrivial totally umbilical hypersurfaces of  $\mathbb{S}^n \times \mathbb{R}$  which are all rotational.

**Theorem 3.** *There exists a one-parameter family  $\{\Sigma(c) ; c > 0\}$  of totally umbilical hypersurfaces of  $\mathbb{S}^n \times \mathbb{R}$  with the following properties:*

- (i) *For any  $c > 0$ ,  $\Sigma(c)$  is complete, analytic, properly embedded and rotational.*
- (ii) *For  $c = 1$ ,  $\Sigma(c)$  is an unbounded graph over an open hemisphere of  $\mathbb{S}^n$ . In particular,  $\Sigma(c)$  is homeomorphic to  $\mathbb{R}^n$ .*



(iii) For  $c \neq 1$ ,  $\Sigma(c)$  is homeomorphic to  $\mathbb{S}^n$ , and is invariant under reflection with respect to a horizontal hyperplane  $\mathbb{S}^n \times \{t_0\}$ .

*Proof.* The hypersurfaces  $\Sigma(c)$  will be obtained by properly “gluing”  $(f_s, \phi)$ -graphs, except for case (ii), in which  $\Sigma(c)$  is an  $(f_s, \phi)$ -graph itself. To accomplish that, let us fix  $o \in \mathbb{S}^n$  and, for each  $s \in (0, \pi/2)$ , let  $f_s : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^n$  be the geodesic sphere of  $\mathbb{S}^n$  with center at  $o$  and radius  $s$ . Then,  $f_s$  is umbilical with principal curvature

$$\lambda(s) = -\cot s$$

with respect to the outward orientation.

Given  $c > 0$ , define the interval  $I_c = [0, a)$ , where  $a \in (0, \pi/2]$  and

$$a = \begin{cases} \pi/2 & \text{if } c = 1 \\ \arcsin(1/c) & \text{if } c > 1 \\ \arcsin(c) & \text{if } c < 1 \end{cases}.$$

In this setting, the function  $\varrho_c : [0, a) \rightarrow \mathbb{R}$  given by

$$\varrho_c(s) = \begin{cases} \sin(s) & \text{if } c = 1 \\ c \sin(s) & \text{if } c > 1 \\ \frac{1}{c} \sin(s) & \text{if } c < 1 \end{cases} \quad (15)$$

is clearly a solution of  $y' = \cot(s)y$  which satisfies

$$0 < \varrho_c|_{(0,a)} < 1, \quad \varrho_c(0) = 0, \quad \text{and} \quad \lim_{s \rightarrow a} \varrho_c(s) = 1. \quad (16)$$

Thus, by Lemma 1,  $\varrho_c$  determines a totally umbilical  $(f_s, \phi)$ -graph  $\Sigma$  which is rotational, for the hypersurfaces  $f_s$  are concentric spheres. Moreover, since  $\varrho$  is real analytic, by (10), so is  $\phi$ , which implies that the graph  $\Sigma$  itself is analytic. In addition,  $\phi(0) = \phi'(0) = 0$ . So  $o \in \Sigma$  is a minimum of the height function of  $\Sigma$ .

For  $c = 1$ , we have  $\varrho_c(s) = \sin(s)$ . In this case, it follows from (10) that

$$\phi(s) = \log \left( \frac{1}{\cos(s)} \right), \quad s \in I_c = [0, \pi/2).$$

In particular,  $\phi(s) \rightarrow +\infty$  as  $s \rightarrow \pi/2$ , so that  $\Sigma(1) := \Sigma$  is an unbounded graph over the hemisphere of  $\mathbb{S}^n$  centered at  $o$  (Fig. 1). This covers (ii).

For  $0 < c \neq 1$ , we have that

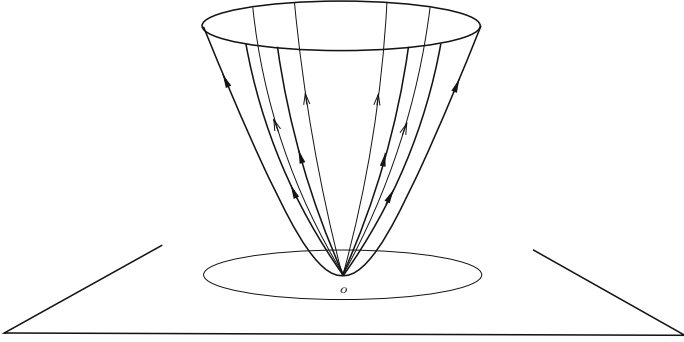
$$\lim_{s \rightarrow a} \varrho'_c(s) = c \cos(a) > 0.$$

So, by Lemma 3,  $\phi$  is bounded, that is, the limit

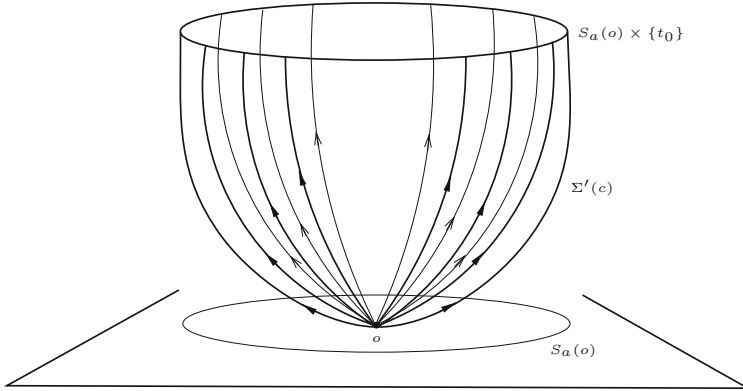
$$t_0 := \lim_{s \rightarrow a} \phi_c(s) = \int_0^a \frac{\varrho_c(s)}{\sqrt{1 - (\varrho_c(s))^2}} ds$$

is well defined. Moreover, from (10) and the last equality in (16), one has

$$\lim_{s \rightarrow a} \phi'(s) = +\infty,$$



**Fig. 1.** An unbounded totally umbilical graph in  $\mathbb{S}^n \times \mathbb{R}$ .



**Fig. 2.** All  $T$ -trajectories of  $\Sigma'(c)$  emanate from  $o$  and meet its boundary  $S_a(o) \times \{t_0\}$  orthogonally

which implies that the tangent spaces of the  $(f_s, \phi)$ -graph  $\Sigma$  along its  $T$ -trajectories converge to vertical tangent spaces along the sphere  $S_a(o) \times \{t_0\}$  (Fig. 2).

Thus, denoting by  $\Sigma'$  the reflection of  $\Sigma$  with respect to  $\mathbb{S}^n \times \{t_0\}$ , we have that

$$\Sigma(c) = \text{closure}(\Sigma) \cup \text{closure}(\Sigma')$$

is a totally umbilical hypersurface of  $\mathbb{S}^n \times \mathbb{R}$ , which is analytic, properly embedded, homeomorphic to  $\mathbb{S}^n$ , and symmetric with respect to  $\mathbb{S}^n \times \{t_0\}$ . This shows (iii) and concludes our proof.  $\square$

**Theorem 4.** *Let  $\Sigma$  be a connected totally umbilical and non totally geodesic hypersurface of  $\mathbb{S}^n \times \mathbb{R}$ . Then,  $\Sigma$  is contained in one of the hypersurfaces  $\Sigma(c)$  given in Theorem 3.*

*Proof.* Since  $\Sigma$  is connected and non totally geodesic, neither  $T$  nor  $\Theta$  can vanish on an open set of  $\Sigma$ . Hence, we can assume that  $\Theta T$  never vanishes on  $\Sigma$ . In this case, it follows from Lemma 2 that  $\Sigma$  is the union of  $(f_s, \phi)$ -graphs  $\Sigma'$  whose

parallel family  $f_s$  is isoparametric and totally umbilical. However, the only parallel family in  $\mathbb{S}^n$  with these properties is that of concentric geodesic spheres, which gives that  $\Sigma'$  is rotational, and that the  $\varrho$ -function of  $\Sigma'$  is one the functions  $\varrho_c$  defined in (15). Therefore,  $\Sigma' \subset \Sigma(c)$ . This fact and the connectedness of  $\Sigma$  clearly imply that  $\Sigma \subset \Sigma(c)$ .

## 6.2. Totally umbilical hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$

The class of complete nontrivial totally umbilical hypersurfaces of  $\mathbb{H}^n \times \mathbb{R}$  is richer than that of  $\mathbb{S}^n \times \mathbb{R}$ . Indeed, as we shall see, besides the rotational ones, this class also includes hypersurfaces which are invariant by certain translational isometries of  $\mathbb{H}^n \times \mathbb{R}$ .

The reason for that abundance becomes clear when we consider, together with Theorem 1, the fact that we have three types of foliations of  $\mathbb{H}^n$  by isoparametric totally umbilical hypersurfaces. Namely, by geodesic spheres, by horospheres, and by equidistant hypersurfaces (from a fixed totally geodesic hyperplane of  $\mathbb{H}^n$ ).

In the next theorems, we show that each of these foliations of  $\mathbb{H}^n$  gives rise to complete nontrivial totally umbilical hypersurfaces of  $\mathbb{H}^n \times \mathbb{R}$ , which are unique, and invariant by one of the aforementioned isometries of this space.

**Theorem 5.** *There exists a one-parameter family  $\{\Sigma(c) ; c > 0\}$  of rotational totally umbilical hypersurfaces of  $\mathbb{H}^n \times \mathbb{R}$  which are complete, analytic and properly embedded. Furthermore, for any  $c > 0$ ,  $\Sigma(c)$  is homeomorphic to  $\mathbb{S}^n$ , and is invariant under reflection with respect to a hyperplane  $\mathbb{H}^n \times \{t_0\}$ .*

*Proof.* Let us fix a point  $o \in \mathbb{H}^n$  and, for each  $s \in (0, +\infty)$ , let  $f_s : \mathbb{S}^{n-1} \rightarrow \mathbb{H}^n$  be the geodesic sphere of  $\mathbb{H}^n$  with center at  $o$  and radius  $s$ . Then,  $f_s$  is umbilical and

$$\lambda(s) = -\coth s$$

is its umbilical function with respect to the outward orientation.

Given  $c > 0$ , define the interval  $I_c = [0, a)$ , where  $a = \sinh^{-1}(1/c)$ . In this setting, it is clearly seen that the function

$$\varrho_c(s) = c \sinh(s), \quad s \in I_c = [0, a),$$

is a solution of  $y' = \coth(s)y$  which satisfies

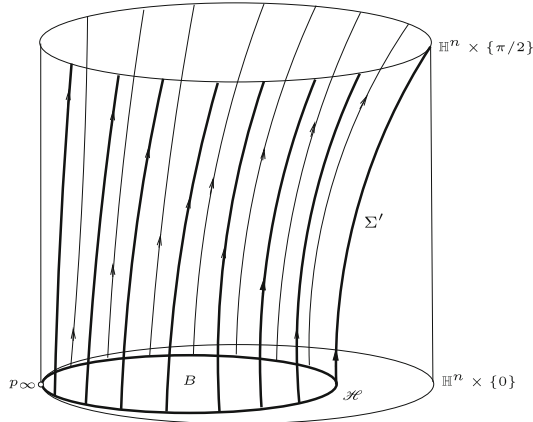
$$0 < \varrho_c|_{(0,a)} < 1, \quad \varrho_c(0) = 0, \quad \text{and} \quad \lim_{s \rightarrow a} \varrho_c(s) = 1. \quad (17)$$

So, by Lemma 1,  $\varrho_c$  determines a totally umbilical  $(f_s, \phi)$ -graph  $\Sigma$  in  $\mathbb{H}^n \times \mathbb{R}$ . As in the proof of Theorem 3,  $\Sigma$  is rotational and analytic. In addition,

$$\lim_{s \rightarrow a} \varrho_c'(s) = c \cosh(a) > 0,$$

which, together with Lemma 3, gives that  $\phi$  is bounded.

Henceforth, arguing just as in the proof of Theorem 3, one easily obtains the desired hypersurface  $\Sigma(c)$  by reflecting  $\Sigma$  with respect to the horizontal hyperplane  $P_{t_0}$ , where  $t_0$  is the limit of  $\phi(s)$  as  $s \rightarrow a$ .  $\square$



**Fig. 3.** All  $T$ -trajectories of  $\Sigma'$  emanate from  $\mathcal{H}$  orthogonally and proceed to the boundary at infinity of the hyperplane  $\mathbb{H}^n \times \{\pi/2\}$

**Theorem 6.** *There exists a complete totally umbilical hypersurface  $\Sigma$  of  $\mathbb{H}^n \times \mathbb{R}$  which is analytic, properly embedded and homeomorphic to  $\mathbb{R}^n$ . Furthermore,  $\Sigma$  is foliated by horizontal horospheres, is symmetric with respect to  $\mathbb{H}^n \times \{0\}$ , is contained in the slab  $\mathbb{H}^n \times (-\frac{\pi}{2}, \frac{\pi}{2})$ , and is asymptotic to the hyperplanes  $P_{-\pi/2}$  and  $P_{\pi/2}$ .*

*Proof.* Let  $\{f_s : \mathbb{R}^{n-1} \rightarrow \mathbb{H}^n ; s \in (-\infty, +\infty)\}$  be a family of parallel horospheres of  $\mathbb{H}^n$ . Each  $f_s$  is totally umbilical with constant umbilical function equal to 1. So, the associated differential equation (11) in this case is  $y' = -y$ . Clearly, the function

$$\varrho(s) = e^{-s}, \quad s \in (0, +\infty),$$

is a solution of this equation which satisfies (see Remark 1 below):

$$0 < \varrho < 1, \quad \lim_{s \rightarrow 0} \varrho(s) = \varrho(0) = 1, \quad \text{and} \quad \lim_{s \rightarrow 0} \varrho'(s) = -\varrho(0) = -1. \quad (18)$$

In addition, from the last equality in (18) and Lemma 3, the function

$$\phi(s) = \int_a^s \frac{\varrho(u)}{\sqrt{1 - (\varrho(u))^2}} du, \quad s \in (0, +\infty),$$

is well defined, i.e., this improper integral is convergent.

Therefore, by Lemma 1, the correspondent  $(f_s, \phi)$ -graph  $\Sigma'$  is totally umbilical in  $\mathbb{H}^n \times \mathbb{R}$ . Notice that, identifying  $\mathbb{H}^n \times \{0\}$  with  $\mathbb{H}^n$ , we have that  $\Sigma'$  is a graph over  $\mathbb{H}^n - B$  with boundary  $\mathcal{H}$ , where  $\mathcal{H}$  is the horosphere  $f_0(\mathbb{R}^{n-1})$  and  $B$  is the horoball bounded by  $\mathcal{H}$  (Fig. 3). Also, since  $\varrho$  is analytic, so is  $\Sigma'$ .

We claim that the function  $\phi$  is bounded above by  $\pi/2$  and that  $\Sigma'$  is asymptotic to the hyperplane  $P_{\pi/2} = \mathbb{H}^n \times \{\pi/2\}$ . Indeed, since  $\varrho' = -\varrho$ , one has

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du = - \int_1^{\varrho(s)} \frac{d\varrho}{\sqrt{1 - \varrho^2}} = \frac{\pi}{2} - \arcsin \varrho(s),$$

which proves the claim, for  $\varrho(s) \rightarrow 0$  as  $s \rightarrow +\infty$ .

Finally, since  $\varrho(0) = 1$ , as in the previous theorems, we have that the tangent spaces of the closure of  $\Sigma'$  along its boundary  $\mathcal{H}$  are all vertical. Consequently, setting  $\Sigma''$  for the reflection of  $\Sigma'$  with respect to  $\mathbb{H}^n \times \{0\}$ , and defining

$$\Sigma := \text{closure}(\Sigma') \cup \text{closure}(\Sigma''),$$

we can argue just as before and conclude that  $\Sigma$  is a complete, analytic, properly embedded, and totally umbilical hypersurface of  $\mathbb{H}^n \times \mathbb{R}$  which is homeomorphic to  $\mathbb{R}^n$ . It is also clear from the construction that  $\Sigma$  is foliated by horizontal horospheres, is symmetric with respect to  $\mathbb{H}^n \times \{0\}$ , and is asymptotic to the hyperplanes  $P_{-\pi/2}$  and  $P_{\pi/2}$ , as we wished to prove.  $\square$

*Remark 1.* In the above proof, taking the general solution  $\varrho_c(s) = ce^{-s}$  of the equation  $y' = -y$ , and proceeding analogously, one obtains a one-parameter family of *isometric* totally umbilical hypersurfaces of  $\mathbb{H}^n \times \mathbb{R}$ , since the horospheres of  $\mathbb{H}^n$  are pairwise isometric. Thus, only the case  $c = 1$  had to be considered.

**Theorem 7.** *There exists a one-parameter family  $\{\Sigma(c); 0 < c < 1\}$  of totally umbilical hypersurfaces of  $\mathbb{H}^n \times \mathbb{R}$  which are analytic, properly embedded and homeomorphic to  $\mathbb{R}^n$ . Furthermore, for any  $c > 0$ ,  $\Sigma(c)$  is foliated by equidistant hypersurfaces, is periodic in the vertical direction, and is symmetric with respect to a discrete set of horizontal hyperplanes.*

*Proof.* Let  $f_0 : \mathbb{R}^{n-1} \rightarrow \mathbb{H}^n$  be a totally geodesic hyperplane of  $\mathbb{H}^n$ , and let

$$\mathcal{F} := \{f_s : \mathbb{R}^{n-1} \rightarrow \mathbb{H}^n; s \in (-\infty, +\infty)\}$$

be the parallel family of equidistant hypersurfaces of  $f_0$  in  $\mathbb{H}^n$ . Each  $f_s$  is totally umbilical with constant umbilical function

$$\lambda(s) = -\tanh(s), \quad s \in (-\infty, +\infty).$$

Thus, the associated differential equation (11) in this case is  $y' = \tanh(s)y$ .

Given  $0 < c < 1$ , let  $a > 0$  be such that  $\cosh(a) = 1/c$ . In this setting,

$$\varrho(s) = c \cosh(s), \quad s \in (-a, a),$$

is a solution of  $y' = \tanh(s)y$  which satisfies

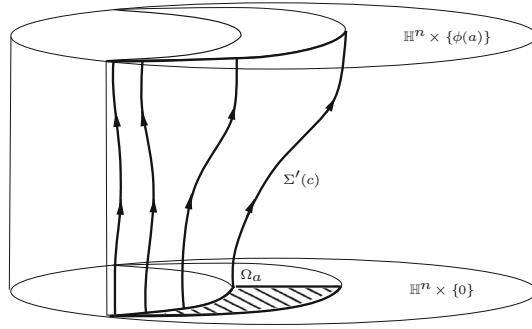
$$0 < \varrho < 1, \quad \lim_{s \rightarrow \pm a} \varrho(s) = \varrho(\pm a) = 1, \quad \text{and} \quad \lim_{s \rightarrow \pm a} \varrho'(s) = c \sinh(\pm a) \neq 0. \quad (19)$$

In addition, from the last equality in (19) and Lemma 3, the function

$$\phi(s) = \int_{-a}^s \frac{\varrho(u)}{\sqrt{1 - (\varrho(u))^2}} du, \quad s \in (-a, a),$$

is well defined and is also bounded.

Therefore, by Lemma 1, the correspondent  $(f_s, \phi)$ -graph  $\Sigma'(c)$  is totally umbilical in  $\mathbb{H}^n \times \mathbb{R}$ . Identifying  $\mathbb{H}^n \times \{0\}$  with  $\mathbb{H}^n$ , we have that  $\Sigma'(c)$  is a graph over the



**Fig. 4.** Half of the graph  $\Sigma'(c)$ . The  $T$ -trajectories emanate from  $\mathcal{E}_{-a} \times \{0\}$  orthogonally, and meet  $\mathcal{E}_a \times \{\phi(a)\}$  orthogonally as well

open convex region  $\Omega_a$  of  $\mathbb{H}^n$  which is bounded by the equidistant hypersurfaces  $\mathcal{E}_{-a} := f_{-a}(\mathbb{R}^{n-1})$  and  $\mathcal{E}_a := f_a(\mathbb{R}^{n-1})$ . In particular,  $\Sigma'(c)$  is homeomorphic to  $\mathbb{R}^n$  and has boundary  $\partial\Sigma'(c) = (\mathcal{E}_{-a} \times \{0\}) \cup (\mathcal{E}_a \times \{\phi(a)\})$  (Fig. 4). Also, as in the preceding theorems,  $\Sigma'(c)$  is analytic, as a consequence of  $\rho$  being analytic.

Now, we have just to observe that, by the second equality in (19), the tangent spaces of  $\Sigma'(c)$  are vertical along its boundary. Therefore, we obtain the hypersurface  $\Sigma(c)$ , as stated, by continuously reflecting  $\Sigma'(c)$  with respect to the horizontal hyperplanes  $\mathbb{H}^n \times \{k\phi(a)\}$ ,  $k \in \mathbb{Z}$ . □

**Theorem 8.** *Let  $\Sigma$  be a connected totally umbilical and non totally geodesic hypersurface of  $\mathbb{H}^n \times \mathbb{R}$ . Then,  $\Sigma$  is contained in one of the complete hypersurfaces given in Theorems 5–7.*

*Proof.* We have just to observe that the only totally umbilical isoparametric hypersurfaces of hyperbolic space  $\mathbb{H}^n$  are geodesic spheres, horospheres, and equidistant hypersurfaces, which are the very horizontal sections of the complete hypersurfaces constructed in Theorems 5–7. Hence, we get to the conclusion by applying the same reasoning as in the proof of Theorem 4. □

In hyperbolic space  $\mathbb{H}^n$ , there are three special types of one-parameter families of isometries. Namely, the rotations around a fixed point (*elliptic isometries*), the translations along a fixed geodesic (*hyperbolic isometries*), and the translations along horocycles sharing the same point at infinity (*parabolic isometries*). It is well known that  $SO(n)$  is the group of elliptic isometries of  $\mathbb{H}^n$ . In the upper half-space model of  $\mathbb{H}^n$ , the hyperbolic isometries are the Euclidean homotheties from the origin, whereas parabolic isometries are horizontal Euclidean translations along fixed horizontal directions (see, e.g., [6, Section 10.1]).

The isometries of  $\mathbb{H}^n$ , in particular those described above, extend to isometries of  $\mathbb{H}^n \times \mathbb{R}$  which fix the factor  $\mathbb{R}$  pointwise. Of course, the same applies to  $\mathbb{S}^n$ , whose isometries are all elliptic. More precisely, given  $\Phi_0 \in \text{Isom}(\mathbb{Q}_\epsilon^n)$ , the map

$$\Phi(p, t) = (\Phi_0(p), t), \quad (p, t) \in \mathbb{Q}_\epsilon^n \times \mathbb{R},$$

is an isometry of  $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ . We call  $\Phi$  *elliptic, parabolic or hyperbolic* according to whether  $\Phi_0$  is elliptic, parabolic or hyperbolic.

An analogous consideration can be made in a warped product  $I \times_\omega \mathbb{Q}_\epsilon^n$ , so that elliptic, parabolic and hyperbolic isometries are also defined in this context.

**Definition 5.** A hypersurface  $\Sigma$  of  $\mathbb{Q}_\epsilon^n \times \mathbb{R}$  or  $I \times_\omega \mathbb{Q}_\epsilon^n$  which is invariant by an elliptic, parabolic or hyperbolic isometry will be called *symmetric*.

Rotational hypersurfaces are obviously invariant by elliptic isometries, which is the case of the hypersurfaces  $\Sigma(c)$  of Theorems 3 and 5. The hypersurface  $\Sigma$  of Theorem 6 is invariant by parabolic isometries, since its horizontal sections project vertically to families of parallel horospheres of  $\mathbb{H}^n$ . Analogously, the hypersurfaces  $\Sigma(c)$  of Theorem 7 are invariant by hyperbolic isometries. These facts, together with Theorems 4 and 8, give the following result.

**Theorem 9.** *All hypersurfaces of Theorems 3, 5, 6 and 7 are symmetric. Consequently, any connected nontrivial totally umbilical hypersurface of  $\mathbb{Q}_\epsilon^n \times \mathbb{R}$  is an open set of a properly embedded symmetric hypersurface.*

We should mention that, in the spirit of the above theorem, a local description of the totally umbilical hypersurfaces of  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$  by means of rotating curves was given in [4, 10].

## 7. Totally umbilical hypersurfaces of $I \times_\omega \mathbb{Q}_\epsilon^n$ .

In what follows, we extend the results of the previous section to the context of warped products  $I \times_\omega \mathbb{Q}_\epsilon^n$ , where  $I \subset \mathbb{R}$  is an open interval. In this setting, let us consider again the conformal diffeomorphism

$$\begin{aligned} \varphi : I \times_\omega \mathbb{Q}_\epsilon^n &\rightarrow \mathbb{Q}_\epsilon^n \times J \\ (t, p) &\mapsto (p, F(t)), \end{aligned} \quad (20)$$

where  $F$  is the diffeomorphism

$$F : I \rightarrow J = F(I), \quad F' = 1/\omega. \quad (21)$$

We will assume, without loss of generality, that

$$J = (-\delta, \delta), \quad 0 < \delta \leq +\infty. \quad (22)$$

As we have already pointed out, the property of being totally umbilical is preserved by the diffeomorphism  $\varphi$ . Furthermore, since the conformal factor of  $\varphi$  depends only on  $t$ , and  $\varphi(\{t\} \times_\omega \mathbb{Q}_\epsilon^n) = \mathbb{Q}_\epsilon^n \times \{F(t)\}$ , we have that  $\{t\} \times_\omega \mathbb{Q}_\epsilon^n$  is homothetic to  $\mathbb{Q}_\epsilon^n \times \{F(t)\}$ . From these considerations, we get the following result.

**Lemma 4.** *Let  $\varphi$  be the conformal diffeomorphism defined in (20). Given a hypersurface  $\Sigma$  in  $I \times_\omega \mathbb{Q}_\epsilon^n$ , the following assertions hold:*

- (i)  $\Sigma$  is totally umbilical in  $I \times_\omega \mathbb{Q}_\epsilon^n$  if and only if  $\varphi(\Sigma)$  is totally umbilical in  $\mathbb{Q}_\epsilon^n \times J$ .

- (ii) A vertical section  $\Sigma_t \subset \Sigma$  is a sphere (resp. horosphere, equidistant hypersurface) of  $\{t\} \times_{\omega} \mathbb{Q}_{\epsilon}^n$  if and only if  $\varphi(\Sigma_t)$  is a horizontal section of  $\varphi(\Sigma)$  which is a sphere (resp. horosphere, equidistant hypersurface) of  $\mathbb{Q}_{\epsilon}^n \times \{F(t)\}$ . Consequently,  $\Sigma$  is symmetric in  $I \times_{\omega} \mathbb{Q}_{\epsilon}^n$  if and only if  $\varphi(\Sigma)$  is symmetric in  $\mathbb{Q}_{\epsilon}^n \times J$ .

Now, we are in position to construct and classify nontrivial totally umbilical hypersurfaces of  $I \times_{\omega} \mathbb{Q}_{\epsilon}^n$ . Firstly, let us recall that the *vertical diameter* of a compact hypersurface of  $M \times \mathbb{R}$  is defined as the difference between the maximum and minimum values of its height function  $\xi$ .

**Theorem 10.** *Given a warping function  $\omega$ , there exists a one-parameter family  $\{\Sigma(c); c > 0\}$  of embedded rotational totally umbilical hypersurfaces of  $I \times_{\omega} \mathbb{Q}_{\epsilon}^n$  with the following properties:*

- (i) For  $\epsilon = 1$  and  $c = 1$ ,  $\Sigma(c)$  is a graph over an open convex geodesic ball of  $\mathbb{S}^n$ . In particular,  $\Sigma$  is homeomorphic to  $\mathbb{R}^n$ .
- (ii) For  $\epsilon = 1$  and  $c \neq 1$ , or  $\epsilon = -1$  and  $c > 0$ ,  $\Sigma(c)$  is either homeomorphic to an  $n$ -annulus or to the  $n$ -sphere  $\mathbb{S}^n$ . The latter occurs if and only if the vertical diameter of  $\varphi(\Sigma(c))$  is less than  $2\delta$ , where  $\delta$  is as in (22).

*Proof.* (i) Let  $\varphi$  be the map defined in (20), and denote by  $\Sigma_0$  the part of the totally umbilical graph of  $\mathbb{S}^n \times \mathbb{R}$  obtained in Theorem 3-(ii) which is contained in  $\mathbb{S}^n \times J$ . Since the whole graph is defined over an open hemisphere of  $\mathbb{S}^n$  centered at a point  $o \in \mathbb{S}^n$ , we have that  $\Sigma_0$  is defined on a maximal convex open ball  $B \subset \mathbb{S}^n$  centered at  $o$ . Therefore, by Lemma 4,  $\Sigma := \varphi^{-1}(\Sigma_0)$  is a rotational totally umbilical graph in  $I \times_{\omega} \mathbb{S}^n$  over  $B$ .

(ii) We consider only the case  $\epsilon = 1$ , since the proof for the case  $\epsilon = -1$  is completely analogous. Let  $\{\Sigma_0(c); c > 0\}$  be the family of totally umbilical spheres of  $\mathbb{S}^n \times \mathbb{R}$  obtained in Theorem 3-(iii). Given  $c > 0$ , after a vertical translation, we can assume that  $\Sigma_0(c)$  is symmetric with respect to the hyperplane  $\mathbb{S}^n \times \{0\}$ . In this way,

$$\Sigma'_0(c) := \Sigma_0(c) \cap (\mathbb{S}^n \times (-\delta, \delta))$$

is clearly homeomorphic to an  $n$ -annulus if the vertical diameter of  $\Sigma_0(c)$  is at least  $2\delta$ . Otherwise,  $\Sigma'_0(c)$  coincides with  $\Sigma_0(c)$ . Therefore,  $\Sigma(c) := \varphi^{-1}(\Sigma'_0(c))$  is an embedded totally umbilical hypersurface of  $I \times_{\omega} \mathbb{S}^n$  which is homeomorphic to an  $n$ -annulus or to the  $n$ -sphere  $\mathbb{S}^n$ . This finishes the proof.  $\square$

**Theorem 11.** *Given a warped product  $I \times_{\omega} \mathbb{H}^n$ , the following hold:*

- (i) *There exists an embedded totally umbilical hypersurface  $\Sigma$  of  $I \times_{\omega} \mathbb{H}^n$  which is foliated by horospheres and is homeomorphic to  $\mathbb{R}^n$ . In addition,  $\Sigma$  is complete, provided that  $\delta > \pi/2$ .*
- (ii) *There exists a one-parameter family  $\{\Sigma(c); 0 < c < 1\}$  of embedded totally umbilical hypersurfaces of  $I \times_{\omega} \mathbb{H}^n$  such that each member  $\Sigma(c)$  is foliated by equidistant hypersurfaces and is homeomorphic to  $\mathbb{R}^n$ .*



*Proof.* We only show (i), since the proof of (ii) is analogous. As in the preceding proof, consider the map  $\varphi$  defined in (20). Denote by  $\Sigma_0$  the part of the totally umbilical hypersurface of  $\mathbb{H}^n \times \mathbb{R}$  obtained in Theorem 6 which is contained in  $\mathbb{H}^n \times (-\delta, \delta)$ . Clearly,  $\Sigma_0$  is homeomorphic to  $\mathbb{R}^n$ . Thus, by Lemma 4, the hypersurface  $\Sigma := \varphi^{-1}(\Sigma_0)$  is embedded and totally umbilical in  $I \times_{\omega} \mathbb{H}^n$ . In addition,  $\Sigma$  is homeomorphic to  $\mathbb{R}^n$  and is foliated by horospheres.

Finally, if  $\delta > \pi/2$ , by Theorem 6,  $\Sigma_0$  is complete and bounded away from the boundary of  $\mathbb{H}^n \times (-\delta, \delta)$ . Thus,  $\Sigma$  is bounded away from the boundary of  $I \times_{\omega} \mathbb{H}^n$ , which implies that it is complete. This finishes the proof.  $\square$

The above results, together with Theorem 9, give the following characterization of totally umbilical hypersurfaces of warped products  $I \times_{\omega} \mathbb{Q}_{\epsilon}^n$ .

**Theorem 12.** *Any connected nontrivial totally umbilical hypersurface of  $I \times_{\omega} \mathbb{Q}_{\epsilon}^n$  is necessarily an open set of an embedded symmetric totally umbilical hypersurface.*

*Remark 2.* It should be mentioned that, through a direct approach, the case  $n = 2$  of Theorem 12 was treated in [3].

To illustrate Theorems 10 and 11, let us consider the warping functions

$$\omega_1(t) = t, \quad t \in I_1 = (0, +\infty), \quad \text{and} \quad \omega_2(t) = e^{-t}, \quad t \in I_2 = (-\infty, +\infty).$$

The associated functions  $F_1, F_2$  satisfying  $F_i' = 1/\omega_i$  are

$$F_1(t) = \log t, \quad t \in I_1, \quad \text{and} \quad F_2(t) = e^t, \quad t \in I_2.$$

Since the range of both  $F_1$  and  $F_2$  is  $\mathbb{R}$ , and  $I_1$  and  $I_2$  are unbounded, the totally umbilical hypersurfaces of  $I_i \times_{\omega_i} \mathbb{Q}_{\epsilon}^n$  from Theorems 10 and 11 are all complete, except for the case (ii) in Theorem 11 applied to  $I_1 \times_{\omega_1} \mathbb{Q}_{\epsilon}^n$ .

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