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A Maximum Principle at Infinity for Surfaces with Constant Mean Curvature in Euclidean Space

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Abstract. Maximum principles at infinity generalize Hopf's maximum principle for hypersurfaces with constant mean curvature in $\mathbb{R}^n$. We establish such a maximum principle for parabolic surfaces in $\mathbb{R}^3$ with nonzero constant mean curvature and bounded Gaussian curvature.

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1. Introduction

One of the classical results in differential geometry is Hopf's maximum principle for surfaces in $\mathbb{R}^3$ with nonzero constant mean curvature. It states that if two such surfaces M_1, M_2 are tangent at an interior point $p \in M_1 \cap M_2$ such that, near p, one lies on one side of the other and, at p, their mean curvature vectors are the same, then M_1 and M_2 coincide in a neighborhood of p. In this case, we will say that M_1 and M_2 have an *ideal contact at* p. By using this result, Alexandrov proved in 1956 that the only compact embedded surfaces in $\mathbb{R}^3$ with constant mean curvature are (round) spheres. This maximum principle also works for minimal surfaces without the assumption on the mean curvature vectors since for such surfaces they equal zero.

Roughly speaking, maximum principles at infinity study the behavior of surfaces with constant mean curvature when the point of contact is at infinity. These principles have been useful in the study of asymptotic ends of minimal and nonzero constant mean curvature surfaces (see Corollary 2, Section 3).

For minimal surfaces, the first result of this kind was proved by Langevin and Rosenberg in [9]:

THEOREM. *Let M_1, M_2 be two disjoint embedded complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature and compact boundaries. Then,* $\operatorname{dist}(M_1, M_2) > 0$.

In [10], Meeks and Rosenberg improved this result by proving the following maximum principle at infinity.

THEOREM. *Let M_1, M_2 be two disjoint properly immersed complete minimal surfaces with nonempty compact boundaries in a complete flat* 3*-manifold. Then,*

$$\mathrm{dist}(M_1, M_2) = \min\{\mathrm{dist}(\partial M_1, M_2), \mathrm{dist}(\partial M_2, M_1)\}.$$

If M_1 and M_2 have empty boundaries then they are flat.

Finally, in [12] Soret studied the case of parabolic minimal surfaces with noncompact boundaries. He obtained the following result:

THEOREM. *Let M_1, M_2 be two disjoint properly embedded complete minimal surfaces with nonempty boundaries that are parabolic in a complete flat* 3*-manifold. Then,*

$$\mathrm{dist}(M_1, M_2) = \min\{\mathrm{dist}(\partial M_1, M_2), \mathrm{dist}(\partial M_2, M_1)\}.$$

In the case one boundary in empty, e.g., M_1, then

$$\mathrm{dist}(M_1, M_2) = \mathrm{dist}(\partial M_2, M_1).$$

Our aim in this paper is to extend Soret's result to parabolic, nonzero constant mean curvature surfaces in $\mathbb{R}^3$ with bounded Gaussian curvature (Theorem 1, Section 3). As in Hopf's maximum principle for nonzero constant mean curvature surfaces, we will need an adittional condition on the mean curvature vectors. It will be called the ideal contact at infinity (see Definition 2 in Section 3).

The paper is organized as follows. In Section 2 we give a brief background on parabolic manifolds. Complete parabolic manifolds without boundary is a rather classic topic in the theory of Riemannian manifolds (see, for example, [6]), however, in this context, manifolds with boundary are given small attention. On this account, we give a definition of parabolicity for such manifolds and prove some results that will be useful later. Finally, in Section 3, we prove the maximum principle at infinity for nonzero constant mean curvature surfaces pointed out above. We will call a surface with constant mean curvature $H \neq 0$ an H-surface and denote the distance function on $\mathbb{R}^3$ by d.

2. Preliminaries

Let $(M^n, \mathrm{d}s^2)$ be an n-dimensional Riemannian manifold with smooth, possibly empty boundary ∂M and metric tensor g_{ij}, that is, in a local chart $(x_1, \ldots, x_n)$ on M one has

$$\mathrm{d}s^2 = \sum_{i,j} g_{ij}\, \mathrm{d}x_i\, \mathrm{d}x_j.$$

The *Laplace operator* Δ on $(M^n, \mathrm{d}s^2)$ is given by

$$\Delta = \mathrm{div}\nabla,$$

where div and ∇ denote, respectively, divergence and gradient on M. In a local chart, the Laplacian is expressed as

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right), \tag{1}$$

where g is the determinant of the matrix (g_{ij}) and $(g^{ij}) = (g_{ij})^{-1}$.

Let $\Omega \subset M$ be an open set in M. A function $u \in C^2(\Omega)$ is called *harmonic* if $\Delta u = 0$. We call a continuous function v on Ω *subharmonic* if, for any relatively compact region $U \subset\subset \Omega$ and any harmonic function $u \in C^2(U) \cap C^0(\bar{U})$, $v \le u$ on ∂U implies $v \le u$ on U. Subharmonic functions satisfy the following maximum principle.

2.1. MAXIMUM PRINCIPLE FOR SUBHARMONIC FUNCTIONS

Suppose $\Omega \subset M$ relatively compact and v a subharmonic function on Ω. Then,

$$\sup_{\Omega} v = \sup_{\partial\Omega} v.$$

It follows easily from this maximum principle that if $v \in C^2(\Omega)$ then the subharmonicity of v is equivalent to $\Delta v \ge 0$.

Suppose ∂M is empty. In this case, M is called *parabolic* if it does not admit a nonconstant subharmonic function bounded above. It is well known that compact manifolds are parabolic. Also, the Euclidean plane $\mathbb{R}^2$ is parabolic and $\mathbb{R}^n$ is not parabolic for $n > 2$. Actually, the parabolicity of $\mathbb{R}^2$ is a consequence of the proposition below, due to Cheng and Yau [2].

PROPOSITION 1. *Let M^n be a geodesically complete Riemannian manifold. Given $p \in M$ and $R > 0$, denote by $V(p, R)$ the volume of the geodesic ball $B_R(p) \subset M$. If, for some point $p \in M$ and for a sequence $R_k \to +\infty$,*

$$V(p, R_k) \le \mathrm{cte.} R_k^2,$$

then M is parabolic.

Proposition 1 also gives that Delaunay surfaces are parabolic. A *Delaunay surface* $\mathcal{D}$ in $\mathbb{R}^3$ is an H-surface of revolution obtained as follows (cf. [3]). Fix a plane Π in $\mathbb{R}^3$ and a line L in Π. Consider an ellipse $\mathcal{E}$ in Π tangent to L and with major axis equal to $1/H$. The generating curve γ of $\mathcal{D}$ is the trace of a chosen focus when one rolls $\mathcal{E}$ on L. Thus, by construction, $\mathcal{D}$ is a (geodesically) complete, periodic surface of revolution. Let s be the arc length of γ and denote by $\rho = \rho(s)$ the distance to the rotation axis of the point $\gamma(s)$. Notice that ρ is bounded for $\mathcal{D}$ is periodic. Given $p \in \mathcal{D}$ we can assume $p = \gamma(0)$. So, for all $R > 0$, the geodesic ball $B_R(p)$ is contained in the part $\mathcal{D}_R$ of $\mathcal{D}$, generated by the segment of γ from $\gamma(-R)$ to $\gamma(R)$. From Pappus' Theorem, the area of $\mathcal{D}_R$ is

$$2\pi \int_{-R}^{R} \rho(s)\, \mathrm{d}(s).$$

Since ρ is bounded, it implies $\mathcal{D}$ has linear area growth, i.e., there is a constant $C > 0$ such that, for all $p \in \mathcal{D}$ and $R > 0$ one has

$$A(p, R) \leq CR.$$

So, by Propostion 1, $\mathcal{D}$ is parabolic.

Parabolic manifolds have a nice property with respect to recurrence of random walks. A complete Riemannian manifold with empty boundary is called *recurrent* if, with respect to the Wiener measure, almost all random walks on the manifold are recurrent, i.e., they return infinitely often to any fixed compact disk. It turns out that parabolicity is equivalent to recurrence (see [6, theorem 5.1]).

On the other hand, a complete Riemannian manifold with nonempty boundary is called *recurrent* if almost all random walks on it hit the boundary. This definition is natural in the sense that a recurrent random walk does not 'escape to infinity'. The following two examples show that, in this case, the definition of parabolicity given above is no longer equivalent to recurrence.

EXAMPLE 1. Let $B \subset \mathbb{R}^2$ be the unit ball in $\mathbb{R}^2$ and $M = \mathbb{R}^2 - B$. Consider the function $v\colon M \to \mathbb{R}$ given by

$$v(p) = \frac{1}{|p|^2}, \quad p \in M.$$

A direct calculation gives $\Delta v \geq 0$, i.e., v is bounded and subharmonic on M. Notice also that $\sup_M v = 1 \equiv v|_{\partial M}$, that is, v satisfies the maximum principle.

EXAMPLE 2. Consider the unit ball B' in $\mathbb{R}^n$, $n > 2$, and let $M' = \mathbb{R}^n - B'$. Define the function $h\colon M' \to \mathbb{R}$ by

$$h(p) = \frac{1}{|p|^{n-2}}, \quad p \in M'.$$

h is easily seen to be bounded and harmonic. In particular the function $v = -h$ is bounded and subharmonic on M'. On the other hand, v does not satisfy the maximum principle for $v|_{\partial M'} \equiv -1$ and $\sup_{M'} v = 0$, that is, $\sup_{M'} v > \sup_{\partial M'} v$.

It can be proved that the manifold M in Example 1 is recurrent whereas M' in Example 2 is not. Therefore, in view of these examples, we will define parabolic manifolds with nonempty boundary as those where the maximum principle for subharmonic functions bounded above is verified. More precisely, DEFINITION 1. A complete n-dimensional Riemannian manifold $(M^n, \mathrm{d}s^2)$ with smooth, nonempty boundary ∂M is called *parabolic* if, for any subharmonic function u on M bounded above one has

$$\sup_M u = \sup_{\partial M} u.$$

Thus, the manifold $M' = \mathbb{R}^n - B'$, $n > 2$, in Example 2 is nonparabolic and, by the maximum principle for subharmonic functions on relatively compact sets, all compact manifolds with boundary are parabolic. In Example 3 below we show the manifold $M = \mathbb{R}^2 - B$ in Example 1 is parabolic.

In what follows, we give some examples of parabolic manifolds with nonempty boundary and prove some results that will be useful in the proof of our main theorem. Throughout the proofs, we shall use the following standard properties of subharmonic functions (see, for example, [1]).

- A finite sum of subharmonic functions is subharmonic, as is any positive scalar multiple of a subharmonic function.
- If u, v are subharmonic on $\Omega \subset M$ then $\max\{u, v\}$ is subharmonic on Ω. In particular, $u_+ = \max\{u, 0\}$ is subharmonic on Ω.

The next proposition gives a sufficient condition for a manifold with boundary to be parabolic, namely, to have a proper, positive harmonic function defined on it. We recall that a function v on a manifold M is called *proper* if, for all compact $K \subset \mathbb{R}$, $v^{-1}(K)$ is compact in M. If M is noncompact this is equivalent to $|v(p)| \to +\infty$ as $p \to +\infty$.

PROPOSITION 2. *Let M be a complete Riemannian manifold with nonempty, smooth boundary ∂M. If there exists a proper, positive harmonic function defined on M, then M is parabolic.*

Proof. Let v be a proper, positive harmonic function on M and consider a subharmonic function u on M, bounded above. Define $\phi = u - \epsilon v$, $\epsilon > 0$. u is bounded and v is proper and positive, hence ϕ attains a global maximum at a point $p \in M$. Since ϕ is clearly subharmonic and nonconstant, p must be a boundary point of M. Therefore, for all $q \in M$,

$$u(q) - \epsilon v(q) \leq u(p) - \epsilon v(p) \leq u(p) \leq \sup_{\partial M} u.$$

By letting ϵ converge to zero we obtain $u(q) \leq \sup_{\partial M} u$. Since $q \in M$ is arbitrary, one has

$$\sup_M u = \sup_{\partial M} u,$$

that is, M is parabolic. □

EXAMPLE 3. Consider $M = \mathbb{R}^2 - B$ as in Example 1. The function $v(p) = \log|p|^2$ is proper, positive and harmonic on M. Then, by Proposition 2, M is parabolic.

EXAMPLE 4. Let C be the cylinder in $\mathbb{R}^3$ given by $C = \{(x_1, x_2, x_3);\ x_1^2 + x_2^2 = 1,\ x_3 \geq 1\}$. Considering the parametrization of C given by $X(\theta, t) = (\cos\theta, \sin\theta, t)$, $0 \leq \theta < 2\pi$, $t \geq 1$, it is easy to see that the function $v(\theta, t) = t$ on C is positive, proper and harmonic. Thus, Proposition 2 gives C parabolic.

More generally, we can consider in $C^n = [1, +\infty) \times \mathbb{S}^{n-1}$ the metric $ds^2 = dt^2 + \sigma(t)^2 d\theta^2$, where dt^2, $d\theta^2$ are the canonical metrics in $[1, +\infty)$ and $\mathbb{S}^{n-1}$, respectively, and σ is a positive, differentiable function on $[1, +\infty)$. The Laplacian on $C^n_\sigma = (C^n, ds^2)$ is given by (see [6])

$$\Delta_\sigma = \frac{\partial^2}{\partial t^2} + (n-1)\frac{\sigma'}{\sigma}\frac{\partial}{\partial t} + \frac{1}{\sigma^2}\Delta_\theta,$$

where Δ_θ denotes the Laplacian on $\mathbb{S}^{n-1}$. By making suitable choices of σ, one obtains examples of parabolic and nonparabolic manifolds.

EXAMPLE 5. Let $\sigma(t) = 1/t$ and consider in C^2_σ the function $v(t, \theta) = t^2/2$. Then,

$$\Delta_\sigma v = \frac{\partial^2 v}{\partial t^2} - \frac{1}{t}\frac{\partial v}{\partial t} + t^2 \Delta_\theta v = 0.$$

Since v is clearly proper and positive, by Proposition 2, C^2_σ is parabolic.
On the other hand, for $\bar\sigma(t) = t^2$, $C^2_{\bar\sigma}$ is nonparabolic. Indeed, the function $u(t, \theta) = -1/t$ on $C^2_{\bar\sigma}$ is subharmonic, bounded and satisfies

$$\sup_{C^2_{\bar\sigma}} u > \sup_{\partial C^2_{\bar\sigma}} u,$$

since $\sup_{C^2_{\bar\sigma}} u = 0$ and $u|_{\partial C^2_{\bar\sigma}} \equiv -1$.

We prove now a proposition that will play a fundamental role in the sequel. It establishes the parabolicity of certain submanifolds of parabolic manifolds. Henceforth, M^n will denote a complete n-dimensional Riemannian manifold with smooth, *possibly empty*, boundary ∂M and $M' \subset M$ a complete embedded n-dimensional Riemannian submanifold of M with *nonempty* smooth boundary $\partial M'$.

LEMMA 1. *Suppose $u \geq 0$ is a subharmonic function on M' such that $u|_{\partial M'} \equiv 0$. Then, the function v on M defined by*

$$v = \begin{cases} u & \text{on } M', \\ 0 & \text{on } M - M', \end{cases}$$

is subharmonic on M.

Proof. Clearly, v is continuous and $v \geq 0$. We must prove that given a relatively compact region $\Omega \subset M$ and a harmonic function h on Ω, $v \leq h$ on $\partial\Omega$ implies $v \leq h$ on Ω. This is obvious for $\Omega \subset M'$ or $\Omega \subset M - M'$ since $v \equiv u$ on M' and v is constant on $M - M'$. Otherwise, $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \Omega \cap M'$, $\Omega_2 = \Omega - \Omega_1$ and $\Omega_i \neq \emptyset$, $i = 1, 2$. On $\partial\Omega$ we have $0 \leq v \leq h$. Thus, $h \geq 0$ on Ω, for h is harmonic. Since $v \equiv 0$ on Ω_2 this gives $v \leq h$ on Ω_2. On the other hand $v \equiv 0 \leq h$ on $\partial\Omega_1 \cap \partial M'$ and $v \leq h$ on $\partial\Omega_1 \cap \partial\Omega$, hence $v \leq h$ on $\partial\Omega_1$. Since $\Omega_1 \subset M'$, $v \equiv u$ on M' and u is subharmonic, this implies $v \leq h$ on Ω_1. Therefore $v \leq h$ on Ω and this proves v is subharmonic on M. □

LEMMA 2. *Suppose ∂M is nonempty. Then M is parabolic if and only if for any bounded subharmonic function $v \geq 0$ on M with $v|_{\partial M} \equiv 0$, one has $v \equiv 0$ on M.*

Proof. The ‘only if’ condition is obvious. Suppose M is nonparabolic, i.e., there exists a subharmonic function u on M, bounded above, such that $\sup_M u > \sup_{\partial M} u$. Let $c \in (\sup_{\partial M} u, \sup_M u)$. Since u is subharmonic, the function $v = (u - c)_+ = \max\{u - c, 0\}$ is subharmonic on M. Furthermore, by the choice of c, v is not identically zero and $v|_{\partial M} \equiv 0$. This proves the ‘if’ assumption of Lemma 2. □

PROPOSITION 3. *M' is parabolic if M is parabolic.*

Proof. Let $v \geq 0$ be a subharmonic function on M', bounded above, such that $v|_{\partial M'} \equiv 0$. By Lemma 2, it suffices to prove that $v \equiv 0$ on M'. Extend v to M by seting $v \equiv 0$ on $M - M'$ and notice that, by Lemma 1, v is subharmonic on M. If ∂M is empty, we must have $v \equiv 0$, otherwise v would be a nonconstant, subharmonic function bounded above on M, which contradicts M is parabolic. If ∂M is nonempty, v satisfies $v|_{\partial M} \equiv 0$ and Lemma 2 gives $v \equiv 0$ on M. □

3. The Maximum Principle at Infinity

As presented in the Introduction, many maximum principles at infinity for minimal surfaces were obtained by several authors. Since Hopf’s maximum principle is valid for minimal and nonzero constant mean curvature surfaces, it seems to be natural the existence of a maximum principle at infinity for nonzero constant mean curvature surfaces. In this section we prove there is such a maximum principle for H-surfaces in $\mathbb{R}^3$ with bounded Gaussian curvature. We also suppose one of the surfaces parabolic and that they have an ideal contact at infinity (see Definition 2 below). When the H-surfaces have bounded Gaussian curvature, the ideal contact at infinity is analougous to the ideal contact at a point as in Hopf’s maximum principle. As we will see, in this case, their mean curvature vectors ‘at infinity’ will be the same (cf. proof of Theorem 1).

DEFINITION 2. Let M_1, M_2 be two disjoint, properly embedded H-surfaces in $\mathbb{R}^3$ such that $d(M_1, M_2) = 0$. We say that M_1 and M_2 have an *ideal contact at*

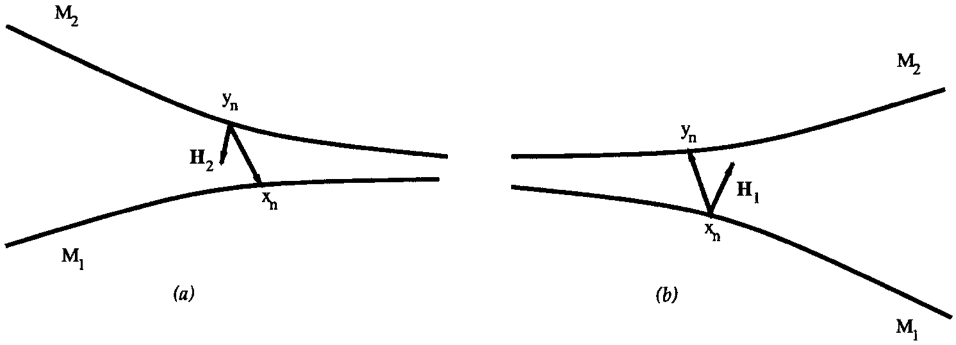

Figure 1. Conditions (a) and (b) of Definition 2.

infinity if there are sequences of interior points $x_n \in M_1$, $y_n \in M_2$, with $|x_n - y_n| \to 0$ as $n \to \infty$ and satisfying, for all n, one of the following two conditions:

(a) $\langle x_n - y_n, \mathbf{H}_2(y_n)\rangle \geq 0$,
(b) $\langle y_n - x_n, \mathbf{H}_1(x_n)\rangle \geq 0$,

where $\mathbf{H}_1, \mathbf{H}_2$ are, respectively, the mean curvature vectors of M_1 and M_2 and $\langle\,,\rangle$ denotes the canonical inner product in $\mathbb{R}^3$ (see Figure 1).

THEOREM 1. *Let M_1, M_2 be two disjoint, complete properly embedded H-surfaces in $\mathbb{R}^3$ with bounded Gaussian curvature and nonempty boundaries ∂M_1, ∂M_2. If M_1 and M_2 have an ideal contact at infinity and either M_1 or M_2 is parabolic then*

$$\min\{d(M_1, \partial M_2), d(M_2, \partial M_1)\} = 0.$$

Proof. Since M_1 and M_2 have an ideal contact at infinity, there exist sequences of interior points $x_n \in M_1$, $y_n \in M_2$ with $|x_n - y_n| \to 0$ as $n \to \infty$ and satisfying (a) or (b) in Definition 2. Without loss of generality, we can assume that (b) is satisfied. First, we shall prove that the mean curvature vectors of M_1 and M_2 point to the same side in the sense that for n large enough, the inner product of these vectors at x_n and y_n, respectively, is positive. Since M_1 and M_2 are H-surfaces with bounded Gaussian curvature, they have bounded second fundamental form. If either $x_n \to \partial M_1$ or $y_n \to \partial M_2$ as $n \to \infty$, we are done. Otherwise, there is a $\delta > 0$ such that, for each $p_1 \in M_1$ and $p_2 \in M_2$, M_1 and M_2 are, locally, graphs over the disks of radius δ in $T_{p_1}M_1$ and $T_{p_2}M_2$, respectively, centered at the origin. Notice that, the second fundamental forms of M_1 and M_2 uniformly bounded gives a C^1-uniform bound for the functions that define these local graphs. Now, fix a point $O \in \mathbb{R}^3$ and, for each n, let φ_n be the translation in $\mathbb{R}^3$ that maps x_n to O. The unit normal vectors of $\varphi_n(M_1)$ at O define a sequence in $\mathbb{S}^2 \subset \mathbb{R}^3$ with a convergent

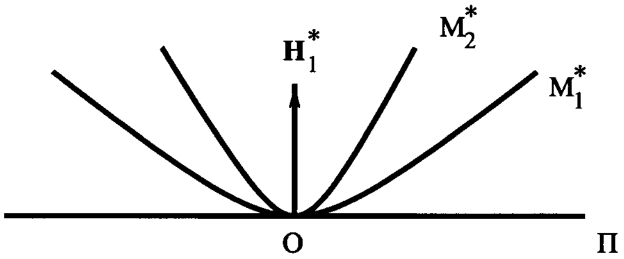

Figure 2.

subsequence. This subsequence determines a subsequence of the sequence of the tangent planes of $\varphi_n(M_1)$ at O that converges to a plane Π_1 in $\mathbb{R}^3$ containing O. Thus, for n large enough, the local graphs of $\varphi_n(M_1)$ at $\varphi_n(x_n) = O$ determine a sequence of H-graphs of functions u_n over the disk $D_{\delta/2} \subset \Pi_1$ centered at O and with radius $\delta/2$. Each u_n is a solution of a quasi-linear partial differential elliptic equation of second order; the mean curvature equation. Since u_n is C^1-uniformly bounded, there is a subsequence u_{n_k} of u_n and a differentiable function u_∞ on $D_{\delta/2}$ such that u_{n_k} converges to u_∞ in the C^∞-topology and the graph of u_∞ has constant mean curvature H [5]. Let M_1^* be the H-graph of u_∞ on $D_{\delta/2}$ and notice that M_1^* is tangent to Π_1 at O. Since the sequence $|x_n - y_n|$ converges to zero as $n \to \infty$, the sequence $\varphi_n(y_n)$ converges to O. Then, applying the same reasoning above to the local graphs of $\varphi_n(M_2)$ at $\varphi_n(y_n)$, we obtain a plane Π_2 in $\mathbb{R}^3$, containing O, and an H-graph M_2^* tangent to Π_2 at O that is the limit of the local graphs of $\varphi_n(M_2)$ at $\varphi_n(y_n)$. But, for all n, $\varphi_n(M_1)$ and $\varphi_n(M_2)$ are disjoint. Hence, we must have $\Pi_1 = \Pi_2$ otherwise the local graphs of $\varphi_n(M_1)$ near M_1^* would intersect the local graphs of $\varphi_n(M_2)$ near M_2^*.

Now, M_1^* and M_2^* are H-graphs of functions f_1, f_2, respectively, over a disk D in a plane Π, both tangent to Π at $O \in \Pi$. Let $\mathbf{H}_1^*$ and $\mathbf{H}_2^*$ be the mean curvature vectors of M_1^* and M_2^* at O, respectively, and η the unit normal vector of Π such that $\langle \mathbf{H}_1^*, \eta \rangle > 0$. M_1 and M_2 are disjoint and the sequences $x_n \in M_1$, $y_n \in M_2$, satisfy the inequality (b) in Definition 2. Hence, the orientation of Π given by η gives $f_2 \geq f_1$. Then, since M_1^* touches M_2^* at O, a comparison principle of mean curvatures at O gives $\langle \mathbf{H}_1^*, \mathbf{H}_2^* \rangle \geq 0$ (see Figure 2). This implies that the mean curvature vectors of M_1 and M_2 point to the same side. Moreover, by Hopf's maximum principle for H-graphs, we conclude that $M_1^* = M_2^*$. This means the local graphs of M_2 at y_n approach asymptotically the local graphs of M_1 at x_n.

We will work now in subsets of M_1 and M_2 which are close to each other. As we will see, on these subsets we can define suitable functions whose gradients are small. This property will imply these functions satisfy an inequality that will permit us to construct subharmonic functions on these subsets. Then, having supposed the theorem is not true, we will derive a contradiction by using the parabolicity assumption.

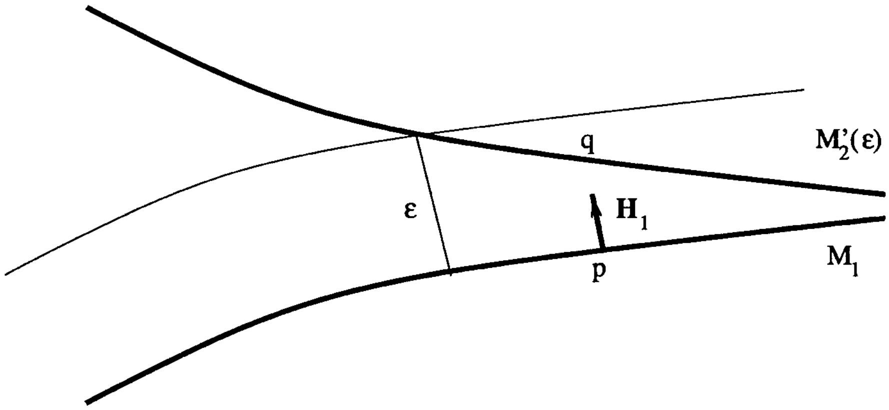

Figure 3.

So, suppose now the theorem fails, i.e.,

$$m_0 = \min\{d(M_1, \partial M_2), d(M_2, \partial M_1)\} > 0.$$

Assume first that M_2 is parabolic. Given $\epsilon > 0$, let

$$M_2(\epsilon) = \{q \in M_2; d(q, M_1) \leq \epsilon\}.$$

For each $q \in M_2(\epsilon)$, consider the set

$$S_q = \{p \in M_1; |p - q| = d(q, M_1) \text{ and } \langle q - p, \mathbf{H}_1(p)\rangle > 0\},$$

and define $M_2'(\epsilon) \subset M_2(\epsilon)$ by

$$M_2'(\epsilon) = \left\{q \in M_2(\epsilon); S_q \text{ is nonempty}\right\}.$$

Notice that M_2' has nonempty interior and is noncompact since M_1 and M_2 are properly embedded and satisfy condition (b) of Definition 2 (see Figure 3). We will show that, for ϵ small, there is a well defined orthogonal projection from $M_2'(\epsilon)$ to M_1, that is, for all $q \in M_2'(\epsilon)$, S_q contains only one point of M_1. Indeed, if it is not true, there is a sequence $q_n \in M_2$ such that $d(q_n, M_1) \to 0$ as $n \to \infty$ and sequences $p_n, p_n' \in M_1$ satisfying, $p_n, p_n' \in S_{q_n}$. Since $d(q_n, M_1) \to 0$, one has $|p_n - p_n'| \to 0$ as $n \to \infty$. Thus, we can repeat the reasoning in the previous paragraphs, replacing the sequences x_n, y_n by the sequences p_n, p_n', respectively. As before, after perform suitable translations, the local graphs of M_1 at p_n and p_n' will converge respectively to graphs G, G', both tangent to a plane $\Pi \subset \mathbb{R}^3$ at a point $O \in \Pi$. Since $p_n, p_n' \in S_{q_n}$, for each n, the mean curvature vectors $\mathbf{H}_1(p_n), \mathbf{H}_1(p_n')$ point to q_n. As a consequence, the mean curvature vectors of G and G' at O point to opposite directons with respect to those determined by the comparison principle of mean curvatures (see Figure 4). This contradiction proves the assertion.

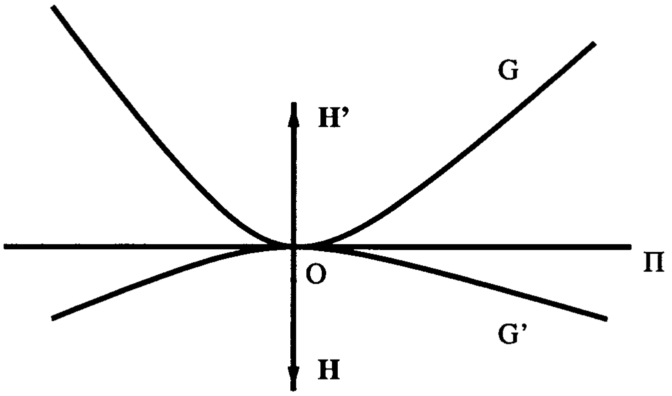

Figure 4.

Therefore, since M_1 and M_2 are embedded, disjoint and have bounded second fundamental forms, if ϵ is small, the orthogonal projection of M_2' to M_1 in $\mathbb{R}^3$,

$$\begin{aligned} \pi_2\colon\ M_2'(\epsilon) &\to M_1 \\ q &\to p \in S_q \end{aligned}$$

is well defined and is a local diffeomorphism onto $\pi_2(M_2'(\epsilon))$.

Let $C_2(\epsilon) \subset M_2{}'(\epsilon)$ be a connected component of $M_2'(\epsilon)$. Since π_2 is a local diffeomorphism, given $q \in C_2(\epsilon)$ and $p = \pi_2(q)$, there are neighborhoods $\Omega_1 \subset \pi_2(C_2(\epsilon))$ and $\Omega_2 \subset C_2(\epsilon)$ of p and q, respectively, and a function $\xi_{\Omega_1} = \xi \in C^\infty(\Omega_1)$, $|\xi| \le \epsilon$, such that Ω_2 is a normal graph of ξ in $\mathbb{R}^3$. This means that, for all $p' \in \Omega_1$ there is one and only one $q' \in \Omega_2$ such that $q' = p' + \xi(p')N(p')$, where N is the unit normal field in M_1 in the orientation given by $\mathbf{H}_1$. With this orientation we have $\xi > 0$ since for all $q' \in \Omega_2$, $\langle q' - \pi_2(q'), \mathbf{H}_1(\pi_2(q'))\rangle > 0$. Notice that the behavior of ξ depends on the magnitude of ϵ for we must have $|\xi| \le \epsilon$. Also, as we have seen, M_1 and M_2 are, locally, graphs of C^1-uniformly bounded functions on δ-disks in their tangent planes. This implies that if the distance between $p \in M_1$ and $q \in M_2$ is small, the tangent planes of M_1 and M_2 at p and q are almost parallel. Moreover, it is easy to see that the angle between T_pM_1 and T_qM_2 is bounded by a constant that depends only on the distance between M_1 and M_2. Therefore, the gradient of ξ converges uniformly to zero as ϵ converges to zero.

Our goal now is to prove that, if ϵ is sufficiently small, the function ξ satisfies an inequality of type $\mathcal{M}\xi < 0$, where $\mathcal{M}$ is a quasi-linear elliptic partial differential operator of second order. This will be done in Lemmas 3 and 4 below. First, it will be convenient to fix some notations.

Let $\mathrm{d}\bar{s}_1^2$ be the metric in $M_1' = \pi_2(M_2'(\epsilon)) \subset M_1$ induced by π_2. Observe that with this metric M_1' is locally isometric to $M_2'(\epsilon)$. Denote by Δ and ∇ (resp. $\bar{\Delta}$ and $\bar{\nabla}$), the Laplacian and the gradient of M_1' (resp. $\left(M_1', \mathrm{d}\bar{s}_1^2\right)$). Let B be the second fundamental form of M_1 and B^* the operator related to B by $B^*B = KI$, where

K is the Gaussian curvature of M_1 and I is the identity operator. Finally, let $\bar{N}$ be the unit vector field in M_2 in the orientation given by its mean curvature vector.

LEMMA 3. *The function ξ satisfies the following equation:*

$$\bar{\Delta}\xi + \alpha^2 \left\{2H - (4H^2 + 2K)\xi + 6HK\xi^2 - 2K^2\xi^3 + \langle B^*\nabla\xi, \nabla\xi\rangle - |B^*\nabla\xi|^2\xi\right\}$$
$$= 2H\alpha(1 - 2H\xi + \xi^2 K), \tag{2}$$

where

$$\alpha = \frac{1}{\sqrt{(1 - 2H\xi + K\xi^2)^2 + |\nabla\xi - \xi B^*\nabla\xi|^2}}.$$

Proof. Let $X = X(u, v)$ be a parametrization of Ω_1 around p and $p_0 = X^{-1}(p)$. We define a parametrization $\bar{X} = \bar{X}(u, v)$ in Ω_2 around q by

$$\bar{X} = X + \xi N.$$

Since M_2 has constant mean curvature H, we have that

$$\bar{\Delta}\bar{X} = 2H\bar{N}.$$

We will get Equation (2) from

$$\langle \bar{\Delta}\bar{X}, N\rangle = 2H\langle \bar{N}, N\rangle. \tag{3}$$

Write E, F, G and e, f, g for the coefficients of the first and second fundamental forms of X, respectively, and let $\bar{E}, \bar{F}, \bar{G}$ and $\bar{e}, \bar{f}, \bar{g}$ be the corresponding objects of $\bar{X}$. To simplify calculations, we will assume the parametrization X is isothermal, i.e.,

$$E = \rho, \quad F = 0, \quad G = \rho,$$

where $\rho = \rho(u, v)$ is a positive, differentiable function. Furthermore, we will suppose that, *at* p, the vectors X_u and X_v are principal directions in T_pM. In this case, at this point, we have

$$e = k_1\rho, \quad f = 0, \quad g = k_2\rho,$$

$$N_u = -k_1 X_u, \quad N_v = -k_2 X_v,$$

where k_1, k_2 are the principal curvatures of M_1. A direct calculation gives,

$$\bar{E} = \rho(1 - k_1\xi)^2 + \xi_u^2, \quad \bar{F} = \xi_u\xi_v, \quad \bar{G} = \rho(1 - k_2\xi)^2 + \xi_v^2.$$

Now, if ϕ is a differentiable function on Ω_1, by identity (1) in Section 2, we have

$$\bar{\Delta}\phi = \frac{1}{\bar{W}}\left\{\frac{\partial}{\partial u}\left(\frac{\bar{G}\phi_u - \bar{F}\phi_v}{\bar{W}}\right) + \frac{\partial}{\partial v}\left(\frac{\bar{E}\phi_v - \bar{F}\phi_u}{\bar{W}}\right)\right\}, \tag{4}$$

where $\bar{W}^2 = \bar{E}\bar{G} - \bar{F}^2$. Since $\langle X_u, N\rangle = \langle X_v, N\rangle = 0$, (4) gives

$$\langle \bar{\Delta} X, N\rangle = \frac{1}{\bar{W}}\left\{\frac{\bar{G}}{\bar{W}}\langle X_{uu}, N\rangle + \frac{\bar{E}}{\bar{W}}\langle X_{vv}, N\rangle\right\} \tag{5}$$

$$= \frac{1}{\bar{W}^2}\left\{\bar{G}e + \bar{E}g\right\}. \tag{6}$$

Thus,

$$\langle \bar{\Delta} X, N\rangle(p_0) = \frac{\rho}{\bar{W}^2}\left\{\bar{G}k_1 + \bar{E}k_2\right\}. \tag{7}$$

Also, $\langle N_u, N\rangle = \langle N_v, N\rangle = 0$ and $\langle N_{uu}, N\rangle = -\langle N_u, N_u\rangle$, $\langle N_{vv}, N\rangle = -\langle N_v, N_v\rangle$. So,

$$\langle N_{uu}, N\rangle(p_0) = -k_1^2\rho,$$

$$\langle N_{vv}, N\rangle(p_0) = -k_2^2\rho.$$

Therefore,

$$\langle \bar{\Delta} N, N\rangle = \frac{1}{\bar{W}}\left\{\frac{\bar{G}}{\bar{W}}\langle N_{uu}, N\rangle + \frac{\bar{E}}{\bar{W}}\langle N_{vv}, N\rangle\right\}, \tag{8}$$

and

$$\langle \bar{\Delta} N, N\rangle(p_0) = \frac{-\rho}{\bar{W}^2}\left\{\bar{G}k_1^2 + \bar{E}k_2^2\right\}. \tag{9}$$

Let N_1, N_2, N_3 denote the coordinates of N in $\mathbb{R}^3$. Since $\sum N_i^2 \equiv 1$, we have

$$\langle \bar{\Delta}(\xi N), N\rangle = \bar{\Delta}\xi + \langle \bar{\Delta} N, N\rangle\xi + \langle \bar{\nabla}\xi, \bar{\nabla}(\textstyle\sum N_i^2)\rangle = \bar{\Delta}\xi + \langle \bar{\Delta} N, N\rangle\xi.$$

Thus,

$$\langle \bar{\Delta}\bar{X}, N\rangle = \langle \bar{\Delta} X, N\rangle + \langle \bar{\Delta}(\xi N), N\rangle = \bar{\Delta}\xi + \langle \bar{\Delta} X, N\rangle + \langle \bar{\Delta} N, N\rangle\xi.$$

From this and Equations (7), (9) we get,

$$\langle \bar{\Delta}\bar{X}, N\rangle(p_0) = \bar{\Delta}\xi + \frac{\rho}{\bar{W}^2}\left\{\bar{G}k_1 + \bar{E}k_2 - (\bar{G}k_1^2 + \bar{E}k_2^2)\xi\right\} \tag{10}$$

$$= \bar{\Delta}\xi + \frac{\rho}{\bar{W}^2}\left\{k_1\left(\rho(1-k_2\xi)^2 + \xi_v^2\right)(1-k_1\xi)\right\} +$$

$$+ \frac{\rho}{\bar{W}^2}\left\{k_2\left(\rho(1-k_1\xi)^2 + \xi_u^2\right)(1-k_2\xi)\right\}. \tag{11}$$

Now notice that, in the coordinate system we are considering,

$$\nabla\xi = \frac{1}{\rho}\left(\xi_u X_u + \xi_v X_v\right).$$

Hence, at p_0,

$$B^*\nabla\xi = \frac{1}{\rho}\left(k_2\xi_u X_u + k_1\xi_v X_v\right).$$

Thus, from (11), we obtain,

$$\langle\bar{\Delta}\bar{X}, N\rangle(p_0) = \bar{\Delta}\xi + \frac{\rho^2}{\bar{W}^2}\left\{2H - (4H^2 + 2K)\xi + 6HK\xi^2 - 2K^2\xi^3\right\} + + \frac{\rho^2}{\bar{W}^2}\left\{\langle B^*\nabla\xi, \nabla\xi\rangle - |B^*\nabla\xi|^2\xi\right\}. \quad (12)$$

Furthermore,

$$\bar{W}^2 = \left\{\rho(1-k_1\xi)^2 + \xi_u^2\right\}\left\{\rho(1-k_2\xi)^2 + \xi_v^2\right\} - \xi_u^2\xi_v^2 \quad (13)$$

$$= \rho^2\left\{(1 - 2H\xi + K\xi^2)^2 + |\nabla\xi - \xi B^*\nabla\xi|^2\right\} \quad (14)$$

$$= \frac{\rho^2}{\alpha^2}. \quad (15)$$

Combining (12) and (15) we get,

$$\langle\bar{\Delta}\bar{X}, N\rangle(p_0) = \bar{\Delta}\xi + \alpha^2\left\{2H - (4H^2 + 2K)\xi + 6HK\xi^2 - 2K^2\xi^3\right\} + + \alpha^2\left\{\langle B^*\nabla\xi, \nabla\xi\rangle - |B^*\nabla\xi|^2\xi\right\}. \quad (16)$$

On the other hand,

$$\bar{N} = \frac{\bar{X}_u \wedge \bar{X}_v}{|\bar{X}_u \wedge \bar{X}_v|} = \frac{\alpha}{\rho}\left(\bar{X}_u \wedge \bar{X}_v\right).$$

But,

$$\bar{X}_u = X_u + \xi_u N + \xi N_u, \quad \bar{X}_v = X_v + \xi_v N + \xi N_v.$$

So,

$$\bar{N}(p_0) = \frac{\alpha}{\rho}\left\{(k_2\xi - 1)\xi_u X_u + (k_1\xi - 1)\xi_v X_v + \rho(1 - 2H\xi + \xi^2 K)N\right\},$$

and,

$$\langle\bar{N}, N\rangle(p_0) = \alpha\{1 - 2H\xi + \xi^2 K\}.$$

Then, from (3) and (16) we easily obtain Equation (2). Since this equation is independent of the parametrization X and p is arbitrary, the lemma follows. □

LEMMA 4. *If $\epsilon > 0$ is sufficiently small, there exists a constant $c_0 > 0$ such that*

$$\bar{\Delta}\xi - c_0|\bar{\nabla}\xi|^2 < 0. \quad (17)$$

Proof. By Lemma 3, ξ satisfies Equation (2). Let us rewrite this equation as

$$L\xi = (P + Q)\xi,$$

where

$$L\xi = \bar{\Delta}\xi + \alpha^2\langle B^*\nabla\xi, \nabla\xi\rangle,$$

$$P\xi = 2H\alpha(1-\alpha) + 2\alpha\left\{2H^2(\alpha - 1) + K\alpha\right\}\xi,$$

$$Q\xi = 2K^2\alpha^2\xi^3 + 2HK\alpha(1 - 3\alpha^2)\xi^2 + \alpha^2|B^*\nabla\xi|^2\xi.$$

First, we shall prove that, for ϵ small enough, there is a constant $c_1 > 0$ such that,

$$\bar{\Delta}\xi - \alpha^2 c_1|\nabla\xi|^2 < 0.$$

Since $\nabla\xi$ converges uniformly to zero as ϵ converges to zero we have,

$$\lim_{\epsilon\to 0}\frac{Q\xi}{\xi} = 0,$$

for H is constant and K and α are bounded. On the other hand, we have

$$\alpha \le \frac{1}{|1 - 2H\xi + K\xi^2|}$$

and α converges to 1 as ϵ converges to zero. But, since $\xi \to 0$ as $\epsilon \to 0$, for ϵ small, one has $|1 - 2H\xi + K\xi^2| = 1 - 2H\xi + K\xi^2$. Considering ξ as a variable, the Taylor expansion of $1/(1 - 2H\xi + K\xi^2)$ in a neighborhood of $\xi = 0$, gives

$$\frac{1}{1 - 2H\xi + K\xi^2} = 1 + 2H\xi + R(\xi),$$

where $\lim_{\epsilon\to 0} R(\xi)/\xi = 0$. Thus, to evaluate $\lim_{\epsilon\to 0} P\xi/\xi$ we can consider α as $1 + 2H\xi$. So, in this case,

$$\lim_{\epsilon\to 0}\frac{P\xi}{\xi} = -(4H^2 - 2K) < 0.$$

This implies that, for ϵ small,

$$L\xi = \bar{\Delta}\xi + \alpha^2\langle B^*\nabla\xi, \nabla\xi\rangle < 0. \tag{18}$$

As given in the proof of Lemma 3, the expression of $\langle B^*\nabla\xi, \nabla\xi\rangle$ in an isothermal parametrization X is

$$\langle B^*\nabla\xi, \nabla\xi\rangle = k_2\frac{\xi_u^2}{\rho} + k_1\frac{\xi_v^2}{\rho} \ge \min\left\{\inf_{M_1}k_1, \inf_{M_1}k_2\right\}|\nabla\xi|^2 \ge -c_1|\nabla\xi|^2,$$

for some constant $c_1 > 0$. Hence, by (18),

$$\bar{\Delta}\xi - \alpha^2 c_1 |\nabla \xi|^2 < 0, \tag{19}$$

as desired.

Considering the local parametrization $\bar{X} = X + \xi N$ given in the proof of Lemma 3, we obtain

$$\begin{aligned} |\bar{\nabla}\xi|^2 &= \frac{1}{\bar{W}^2}\left\{\bar{G}\xi_u^2 - 2\bar{F}\xi_u\xi_v + \bar{E}\xi_v^2\right\} \\ &= \alpha^2\left\{(1-k_2\xi)^2\frac{\xi_u^2}{\rho} + (1-k_1\xi)^2\frac{\xi_v^2}{\rho}\right\} \geq \alpha^2\beta|\nabla\xi|^2, \end{aligned}$$

where $\beta = \min\left\{\inf_{M_1}(1-k_1\xi)^2, \inf_{M_1}(1-k_2\xi)^2\right\}$. Since k_1 and k_2 are bounded, for ϵ small, one has $\beta > 0$. Thus, by taking $c_0 = c_1/\beta$, we obtain

$$c_0|\bar{\nabla}\xi|^2 \geq \alpha^2 c_1 |\nabla\xi|^2.$$

From this last inequality and (19) we get (17). This proves Lemma 4. □

Now, consider $\epsilon > 0$ such that the inequality (17) in Lemma 4 is satisfied and $m_0 > \epsilon$. From this last assumption we have $\partial M_2 \cap C_2(\epsilon) = \emptyset$, that is, $\partial C_2(\epsilon) = \{q \in C_2(\epsilon); d(q, M_1) = \epsilon\}$. Let $\bar{\xi}$ be the function on $C_2(\epsilon)$ defined by

$$\bar{\xi}(q) = d(q, M_1), q \in C_2(\epsilon)$$

and notice that $\bar{\xi}|_{\partial C_2(\epsilon)} \equiv \epsilon$. Moreover, locally, $\bar{\xi}$ is given by

$$\bar{\xi} = \xi \circ \pi_2.$$

Thus, since $\left(M_1', \mathrm{d}\bar{s}_1^2\right)$ is locally isometric, via π_2, to $M_2'(\epsilon)$, by Lemma 4, $\bar{\xi}$, satisfies

$$\bar{\Delta}\bar{\xi} - c_0|\bar{\nabla}\bar{\xi}|^2 < 0. \tag{20}$$

Observe that $C_2(\epsilon)$ is not compact, otherwise $\bar{\xi}$ would attain a minimum at an interior point $q' \in C_2(\epsilon)$. In this case, $\bar{\nabla}\bar{\xi}(q') = 0$ and, by Equation (20), $\bar{\Delta}\bar{\xi}(q') < 0$, which contradicts that q' is a minimum interior point for $\bar{\xi}$. Therefore $C_2(\epsilon)$ is noncompact and $\sup_{C_2(\epsilon)} \bar{\xi} = \epsilon$.

Consider the function φ on $C_2(\epsilon)$ given by

$$\varphi = \mathrm{e}^{-c_0\bar{\xi}}.$$

A direct calculation gives

$$\bar{\Delta}\varphi = -c_0\,\mathrm{e}^{-c_0\bar{\xi}}\left(\bar{\Delta}\bar{\xi} - c_0|\bar{\nabla}\bar{\xi}|^2\right) > 0,$$

i.e., φ is subharmonic on $C_2(\epsilon)$. Since we have assumed M_2 parabolic, Proposition 3 gives $C_2(\epsilon)$ parabolic. Thus,

$$\sup_{C_2(\epsilon)} \varphi = \sup_{\partial C_2(\epsilon)} \varphi = \mathrm{e}^{-c_0\epsilon},$$

but this is a contradiction for $\sup_{C_2(\epsilon)} \varphi = 1 > \mathrm{e}^{-c_0\epsilon}$. Therefore we must have $m_0 = 0$. This proves Theorem 1 when M_2 is parabolic.

Suppose now M_1 is parabolic. We proceed as before, i.e., given a small $\epsilon > 0$ we define the sets $C_1(\epsilon) \subset M_1'(\epsilon) \subset M_1(\epsilon)$ as we did for M_2. Namely,

$$M_1(\epsilon) = \{p \in M_1; d(p, M_2) \leq \epsilon\},\ M_1'(\epsilon) = \{p \in M_1(\epsilon); S_p \text{ is nonempty}\}$$

and $C_1(\epsilon)$ a connected component of $M_1'(\epsilon)$. Here,

$$S_p = \{q \in M_2; |p - q| = d(p, M_2) \text{ and } \langle q - p, \mathbf{H}_2(q)\rangle > 0\}.$$

For ϵ small, we also have a well defined orthogonal projection π_1: $M_1'(\epsilon) \to M_2$. Observe that by Proposition 3, $C_1(\epsilon)$ is parabolic.

Analogously, we can cover $C_1(\epsilon)$ and $\pi_1(C_1(\epsilon)) \subset M_2$ by neighborhoods $\Lambda_1 \subset C_1(\epsilon)$, , $\Lambda_2 \subset \pi_1(C_1(\epsilon))$, such that Λ_1 is a normal graph (in the orientation given by $\mathbf{H}_2$) of a function $\eta_{\Lambda_2} \equiv \eta \in C^\infty(\Lambda_2), |\eta| \leq \epsilon$. The only essential difference between ξ and η is that $\xi > 0$ whereas $\eta < 0$. Obviously, η satisfies an equation similar to (2). Also, reasoning as in the proof of Lemma 4, we obtain a constant $a_0 > 0$ such that

$$\tilde{\Delta}\eta + a_0|\tilde{\nabla}\eta|^2 > 0,$$

where $\tilde{\Delta}$, $\tilde{\nabla}$ denote, respectively, the Laplacian and the gradient on $\pi_1(C_1(\epsilon))$ in the metric induced by π_1 (to see this, just change ξ by $-\xi$ in (17)). Thus, the function $\tilde{\eta}$ on $C_1(\epsilon)$ given by

$$\tilde{\eta}(p) = -d(p, M_2), \quad p \in C_1(\epsilon)$$

satisfies

$$\tilde{\Delta}\tilde{\eta} + a_0|\tilde{\nabla}\tilde{\eta}|^2 > 0. \tag{21}$$

So, reasoning as before, we have that $C_1(\epsilon)$ is not compact and $\sup_{C_1(\epsilon)} \tilde{\eta} = 0$. Notice also that $\tilde{\eta}|_{\partial C_1(\epsilon)} \equiv -\epsilon$.

Now we define on $C_1(\epsilon)$ the function

$$\phi = \mathrm{e}^{a_0\tilde{\eta}}.$$

From (21)

$$\tilde{\Delta}\phi = a_0\mathrm{e}^{a_0\tilde{\eta}}\left(\tilde{\Delta}\tilde{\eta} + a_0|\tilde{\nabla}\tilde{\eta}|^2\right) > 0,$$

that is, ϕ is subharmonic on $C_1(\epsilon)$. But this contradicts $C_1(\epsilon)$ is parabolic for

$$\sup_{C_1(\epsilon)} \phi = 1 \quad \text{and} \quad \phi|_{\partial C_1(\epsilon)} = \mathrm{e}^{-a_0\epsilon} < 1.$$

Thus, if M_1 is parabolic we also have $m_0 = 0$. This completes the proof of Theorem 1. □

Remark 1. Since Proposition 3 is also valid when the ambient manifold M has empty boundary, the proof above also shows that, under the assumptions of Theorem 1, if one of the surfaces has empty boundary, e.g. M_1, then $d(M_1, \partial M_2) = 0$.

Let M be a complete properly embedded H-surface in $\mathbb{R}^3$ with empty boundary. Then, M divides $\mathbb{R}^3$ into two connected components. We recall that the *mean convex side* of M is the component to which the mean curvature vector of M points to.

A result of Meeks and Rosenberg in [11] shows that a complete embedded H-surface in $\mathbb{R}^3$, with empty boundary and bounded Gaussian curvature is properly embedded. Thus, the proof of Theorem 1 actually yields

COROLLARY 1. *Let M_1, M_2 be two disjoint, complete embedded H-surfaces in $\mathbb{R}^3$ with bounded Gaussian curvature and empty boundaries. Then, if either M_1 or M_2 is parabolic, one surface cannot lie in the mean convex side of the other.*

Proof. Suppose M_2 is in the mean convex side of M_1. If $d(M_1, M_2) = 0$ it implies M_1 and M_2 have an ideal contact at infinity. In this case, we can repeat the reasoning of the proof of Theorem 1 (not considering assumptions on ∂M_1 or ∂M_2) and obtain the same contradictions.

If $d(M_1, M_2) > 0$, there are sequences $p_n \in M_1$ and $q_n \in M_2$ such that the sequence $p_n - q_n$ has a subsequence that converges to a vector $v \in \mathbb{R}^3$ with $|v| = d(M_1, M_2)$. Replacing M_2 by $\overline{M_2} = M_2 + v$ we have $d(M_1, \overline{M_2}) = 0$. By previous considerations, M_1 and $\overline{M_2}$ cannot be disjoint. Thus, there exists $p \in \mathbb{R}^3$ such that $p \in M_1 \cap \overline{M_2}$. Since $|v| = d(M_1, M_2)$ and M_2 is in the mean convex side of M_1, M_1 and $\overline{M_2}$ have an ideal contact at p. By Hopf's maximum principle, M_1 and $\overline{M_2}$ coincide in a neighborhood of p and hence, by analycity, $M_1 = \overline{M_2}$. This means M_1 differs from M_2 by a translation in $\mathbb{R}^3$ of magnitude $d(M_1, M_2)$. Since a segment in $\mathbb{R}^3$ with endpoints in M_1 and M_2 and length $d(M_1, M_2)$ is orthogonal to M_1 and M_2, this implies M_1 and M_2 are parallel planes. This contradicts M_1 and M_2 are H-surfaces and ends the proof of Corollary 1. □

Remark 2. During the preparation of this paper, the author was informed that Ros and Rosenberg have proved a stronger version of Corollary 1.

Let $\mathcal{A} \subset \mathbb{R}^3$ be a properly embedded annulus in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$. In [7], Korevaar et al. prove that $\mathcal{A}$ has bounded Gaussian curvature and there is a Delaunay surface $\mathcal{D}_{\mathcal{A}} \subset \mathbb{R}^3$, with constant mean curvature H, to which $\mathcal{A}$ converges exponentially as $|p| \to +\infty$, $p \in \mathcal{A}$. Since a Delaunay surface is parabolic (cf. Section 2) and has bounded Gaussian curvature, Theorem 1 also gives the following corollary:

COROLLARY 2. *If $\mathcal{A}$ is in the closure of the mean convex side of $\mathcal{D}_{\mathcal{A}}$ and $\partial\mathcal{A}$ is disjoint from $\mathcal{D}_{\mathcal{A}}$, then $\mathcal{A}$ is an end of $\mathcal{D}_{\mathcal{A}}$.*

Proof. Suppose there exists $p \in \mathcal{A} \cap \mathcal{D}_{\mathcal{A}}$. Since $\partial\mathcal{A}$ and $\mathcal{D}_{\mathcal{A}}$ are disjoints, p is an interior point of $\mathcal{A}$. Thus, we can apply Hopf's maximum principle to conclude that $\mathcal{A}$ is an end of $\mathcal{D}_{\mathcal{A}}$.

If $\mathcal{A} \cap \mathcal{D}_{\mathcal{A}}$ is empty, $\mathcal{A}$ and $\mathcal{D}_{\mathcal{A}}$ have an ideal contact at infinity for $\mathcal{A}$ converges exponentially to $\mathcal{D}_{\mathcal{A}}$. Therefore, by Theorem 1 (see also Remark 1) one has $d(\partial\mathcal{A}, \mathcal{D}_{\mathcal{A}}) = 0$. But this contradicts $\mathcal{A}$ and $\mathcal{D}_{\mathcal{A}}$ are disjoints, since $\partial\mathcal{A}$ is compact. □

Remark 3. Corollary 2 is a particular case of a theorem obtained by Earp and Rosenberg in [4].

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