Received: 17 January 2022 | Revised: 1 September 2022 | Accepted: 5 December 2022

DOI: 10.1002/mana.202200025

MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

Elliptic Weingarten hypersurfaces of Riemannian products

R. F. de Lima[1] | **A. K. Ramos**[2] | **J. P. dos Santos**[3]

[1]Departamento de Matemática, UFRN, Natal, Rio Grande do Norte, Brazil

[2]Departamento de Matemática Pura e Aplicada, UFRGS, Natal, Rio Grande do Norte, Brazil

[3]Departamento de Matemática, UnB, Natal, Rio Grande do Norte, Brazil

Correspondence
R. F. de Lima, Departamento de Matemática, UFRN, Brazil.
Email: ronaldo.freire@ufrn.br

Present address
Universidade Federal do Rio Grande do Norte, Centro de Ciências Exatas, Departamento de Matemática. 59000-000 - Natal, RN - Brazil

Funding information
Conselho Nacional de Desenvolvimento Científico e Tecnológico

Abstract

Let M^n be either a simply connected space form or a rank-one symmetric space of the noncompact type. We consider *Weingarten hypersurfaces* of $M \times \mathbb{R}$, which are those whose principal curvatures $k_1, \ldots, k_n$ and angle function Θ satisfy a relation $W(k_1, \ldots, k_n, \Theta^2) = 0$, being W a differentiable function which is symmetric with respect to $k_1, \ldots, k_n$. When $\partial W/\partial k_i > 0$ on the positive cone of $\mathbb{R}^n$, a strictly convex Weingarten hypersurface determined by W is said to be *elliptic*. We show that, for a certain class of Weingarten functions W, there exist rotational strictly convex Weingarten hypersurfaces of $M \times \mathbb{R}$ which are either topological spheres or entire graphs over M. We establish a Jellett–Liebmann-type theorem by showing that a compact, connected and elliptic Weingarten hypersurface of either $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$ is a rotational embedded sphere. Other uniqueness results for complete elliptic Weingarten hypersurfaces of these ambient spaces are obtained. We also obtain existence results for constant scalar curvature hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ which are either rotational or invariant by translations (parabolic or hyperbolic). We apply our methods to give new proofs of the main results by Manfio and Tojeiro on the classification of constant sectional curvature hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.

KEYWORDS
constant scalar curvature, invariant hypersurfaces, Riemannian product, Weingarten hypersurface

MSC (2020)
53B25 (primary), 53C42 (secondary)

1 | INTRODUCTION

Among the compact surfaces of Euclidean space $\mathbb{R}^3$, round spheres are known to be unique with respect to several types of curvature constraints. For instance, they are the only compact connected embedded surfaces which have either constant mean curvature or constant Gaussian curvature, as attested by the classical theorems of Alexandrov and Hilbert–Liebmann, respectively. More generally, these theorems apply to *elliptic Weingarten surfaces*, which are those whose principal curvature functions k_1, k_2 satisfy a relation

$$W(k_1, k_2) = 0,$$

where W is a symmetric differentiable function on a domain $\Gamma \subset \mathbb{R}^2$, which satisfies the ellipticity condition $\partial W/\partial k_i > 0$ on $W^{-1}(0) \subset \Gamma$. In [17], Gálvez and Mira improved these results by showing that round spheres are the only elliptic Weingarten spheres *immersed* in $\mathbb{R}^3$, which proved affirmatively a long standing conjecture by Alexandrov.

Elliptic Weingarten surfaces (sometimes called *special Weingarten* surfaces) of $\mathbb{R}^3$ and other three-spaces have been considered in many works [5, 16, 20, 25]. More recently, Gálvez and Mira [18] conducted a thorough investigation of rotationally invariant elliptic Weingarten surfaces of homogeneous three-manifolds with isometry group of dimension 4 (which include the Riemannian products $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$). There, they established many deep results regarding existence and uniqueness of certain rotational spheres in the class of elliptic Weingarten surfaces of these three-manifolds.

Inspired by the work of Gálvez and Mira, in this paper we consider *Weingarten hypersurfaces* of Riemannian products $M^n \times \mathbb{R}$. They are defined here as those whose principal curvatures $k_1, \dots, k_n$ and angle function Θ satisfy

$$W(k_1, \dots, k_n, \Theta^2) = 0,$$

where W is a differentiable function which is symmetric with respect to $k_1, \dots, k_n$. Such a hypersurface Σ is then called a W-hypersurface of $M \times \mathbb{R}$. If, in addition, Σ is strictly convex and W satisfies the ellipticity condition $\partial W/\partial k_i > 0$ (for all $i = 1, \dots, n$) on the positive cone of $\mathbb{R}^n$, we say that Σ is an *elliptic Weingarten hypersurface.*

Our work primarily concerns existence and uniqueness (in the elliptic case) of Weingarten hypersurfaces of $M \times \mathbb{R}$ when M is either a simply connected space form or a rank-one symmetric space of the noncompact type (i.e., one of the hyperbolic spaces $\mathbb{H}^m_{\mathbb{F}}$). The reason for considering these particular manifolds relies on the fact that their geodesic spheres are isoparametric, that is, have constant principal curvatures. As we shall see, this property allows us to construct Weingarten vertical graphs in $M \times \mathbb{R}$ whose level hypersurfaces are concentric geodesic spheres of M (we call such graphs *rotational*).

Given a general Weingarten function W, we associate with it a first-order ODE which involves the principal curvature functions of geodesic spheres of M. Then, we call W *M-admissible* if this equation admits a solution $\varrho : [0, \delta) \to [0, 1)$ satisfying certain conditions (see Definition 5.1). Then, we show that, for such an M-admissible W, there exists a rotational complete W-hypersurface Σ of $M \times \mathbb{R}$ which is homeomorphic to either the n-sphere $\mathbb{S}^n$ or Euclidean space $\mathbb{R}^n$. In the latter case (which does not occur if $M = \mathbb{S}^n$), Σ is an entire graph over M, and in the former case, Σ is obtained from the connected sum of a graph over a closed ball of M with its reflection over a horizontal hyperplane of $M \times \mathbb{R}$.

Next, we study constant scalar curvature (CSC) hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, where $n \geq 3$ and $\mathbb{Q}^n_\epsilon$ denotes the simply connected space form $\mathbb{Q}^n_\epsilon$ of constant sectional curvature $\epsilon = \pm 1$ (i.e., $\mathbb{S}^n$ and $\mathbb{H}^n$). Based on the fact that such hypersurfaces are Weingarten, we establish that, for all $c > \epsilon n(n-1)$, there exists a properly embedded strictly convex rotational hypersurface Σ in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with CSC c, which is, in fact, of constant sectional curvature. Such a Σ is either a sphere (if $c > 0$) or an entire graph (if $c \leq 0$). For $c > \epsilon n(n-1)$, we also obtain a one-parameter family of properly embedded Delaunay-type rotational n-annuli in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with CSC c, which are not of the constant sectional curvature. An analogous one-parameter family of nonperiodic rotational n-annuli in $\mathbb{H}^n \times \mathbb{R}$ is obtained as well.

Similar results hold for hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ which are invariant by either parabolic or hyperbolic translations. Specifically, we show that for any constant $c \in [-n(n-1), 0)$, there exist in $\mathbb{H}^n \times \mathbb{R}$ an entire graph over $\mathbb{H}^n$ (of constant sectional curvature) and a hypersurface which is symmetric with respect to a horizontal hyperplane, both of the CSC c and invariant by parabolic translations. For such values of c, we also show that there exists a one-parameter family of hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ with CSC c which are invariant by hyperbolic translations. All these translational hypersurfaces are properly embedded and homeomorphic to $\mathbb{R}^n$.

As these results indicate, hypersurfaces of the constant sectional curvature appear naturally when we are dealing with CSC hypersurfaces. Considering this fact, we apply the methods developed here to provide new proofs for the main theorems by Manfio and Tojeiro [22] regarding the classification of hypersurfaces of constant sectional curvature of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. (On this matter, see also [2, 3], where the case $n = 2$ was considered.)

Regarding uniqueness of elliptic Weingarten hypersurfaces, we establish a Jellett–Liebmann-type theorem. Namely, by means of the maximum principle, the Alexandrov reflection technique, and the methods and results in [7, 13–15, 23], we show that, for $n \geq 3$, a compact connected strictly convex elliptic Weingarten hypersurface immersed in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ is necessarily an embedded rotational sphere. It is also shown that, assuming such a hypersurface to be complete, instead of compact, the same conclusion holds under the additional assumption that its height function has a critical point, and that its least principal curvature is bounded away from zero.

The paper is organized as follows. In Section 2, we set some notation and formulae. In Section 3, we discuss graphs of Riemannian products $M \times \mathbb{R}$ over parallel hypersurfaces of M. In Section 4, we introduce Weingarten hypersurfaces of

$M \times \mathbb{R}$, establishing a key lemma. In Section 5, we consider rotational Weingarten hypersurfaces of $\mathbb{H}_{\mathbb{F}}^m \times \mathbb{R}$ and $\mathbb{Q}_\epsilon^n \times \mathbb{R}$. In Section 6, we deal with CSC hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ which are invariant by either rotational or translational isometries. In Section 7, we establish the Jellett–Liebmann-type theorem we mentioned, together with other rigidity results for Weingarten hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$. In Section 8, we consider hypersurfaces of the constant sectional curvature of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$.

2 PRELIMINARIES

Given an orientable Riemannian manifold M^n, $n \geq 2$, we shall consider the Riemannian product $M \times \mathbb{R}$ endowed with its standard metric

$$\langle \, , \, \rangle = \langle \, , \, \rangle_M + dt^2.$$

Let Σ be an oriented hypersurface of $M \times \mathbb{R}$. Write N for its unit normal field and A for its shape operator with respect to N, so that

$$AX = -\overline{\nabla}_X N, \; X \in T\Sigma,$$

where $\overline{\nabla}$ denotes the Levi–Civita connection of $M \times \mathbb{R}$ and $T\Sigma$ stands for the tangent bundle of Σ. The principal curvatures of Σ, that is, the eigenvalues of the shape operator A, will be denoted by $k_1, \dots, k_n$. We shall say that Σ is *convex* (resp. *strictly convex*) if, with respect to a suitable orientation N, $k_i \geq 0$ (resp. $k_i > 0$) everywhere on Σ, $i = 1, \dots, n$.

The *height function* ξ and the *angle function* Θ of Σ are defined as

$$\xi := \pi_{\mathbb{R}}|_\Sigma \quad \text{and} \quad \Theta(x) := \langle N(x), \partial_t \rangle, \; x \in \Sigma,$$

where ∂_t denotes the gradient of the projection $\pi_{\mathbb{R}}$ of $M \times \mathbb{R}$ on the second factor $\mathbb{R}$. We denote the gradient of ξ on Σ by T, which means that the equality

$$T = \partial_t - \Theta N \tag{2.1}$$

holds on Σ. The trajectories of T on Σ will be called *T-trajectories.*

Given $t \in \mathbb{R}$, the set $P_t := M \times \{t\}$ is called a *horizontal hyperplane* of $M \times \mathbb{R}$. Horizontal hyperplanes are all isometric to M and totally geodesic in $M \times \mathbb{R}$. In this context, we call a transversal intersection $\Sigma_t := \Sigma \pitchfork P_t$ a *horizontal section* of Σ. Any horizontal section Σ_t is a hypersurface of P_t. So, at any point $x \in \Sigma_t \subset \Sigma$, the tangent space $T_x\Sigma$ of Σ at x splits as the orthogonal sum

$$T_x\Sigma = T_x\Sigma_t \oplus \mathrm{Span}\{T\}. \tag{2.2}$$

The (first factor) manifolds M we shall consider here are the simply connected space forms $\mathbb{Q}_\epsilon^n$ of the constant sectional curvature $\epsilon \in \{0, 1, -1\}$, that is, the Euclidean space $\mathbb{R}^n$, the unit sphere $\mathbb{S}^n$ ($\epsilon = 1$) and the hyperbolic space $\mathbb{H}^n$ ($\epsilon = -1$), as well as the rank-one symmetric spaces of noncompact type, also known as the general hyperbolic spaces $\mathbb{H}_{\mathbb{F}}^m$. Note that the real hyperbolic space $\mathbb{H}_{\mathbb{R}}^m$ is the standard hyperbolic space $\mathbb{H}^n$ of constant sectional curvature -1 (see, e.g., [12]).

Let us recall that, denoting by R and $\overline{R}$ the curvature tensors of Σ and $\mathbb{Q}_\epsilon^n \times \mathbb{R}$, respectively, for $X, Y, Z, W \in T\Sigma$, the Gauss equation reads as

$$\langle R(X,Y)Z, W \rangle = \langle \overline{R}(X,Y)Z, W \rangle + \langle AX, W \rangle \langle AY, Z \rangle - \langle AX, Z \rangle \langle AY, W \rangle, \tag{2.3}$$

where $\overline{R}$ vanishes identically for $\epsilon = 0$ and, for $\epsilon = \pm 1$, it is given by (see [6])

$$\begin{aligned} \epsilon \langle \overline{R}(X,Y)Z, W \rangle = {} & \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \\ & + \langle X, Z \rangle \langle Y, \partial_t \rangle \langle W, \partial_t \rangle - \langle Y, Z \rangle \langle X, \partial_t \rangle \langle W, \partial_t \rangle \\ & - \langle X, W \rangle \langle Y, \partial_t \rangle \langle Z, \partial_t \rangle + \langle Y, W \rangle \langle X, \partial_t \rangle \langle Z, \partial_t \rangle. \end{aligned} \tag{2.4}$$

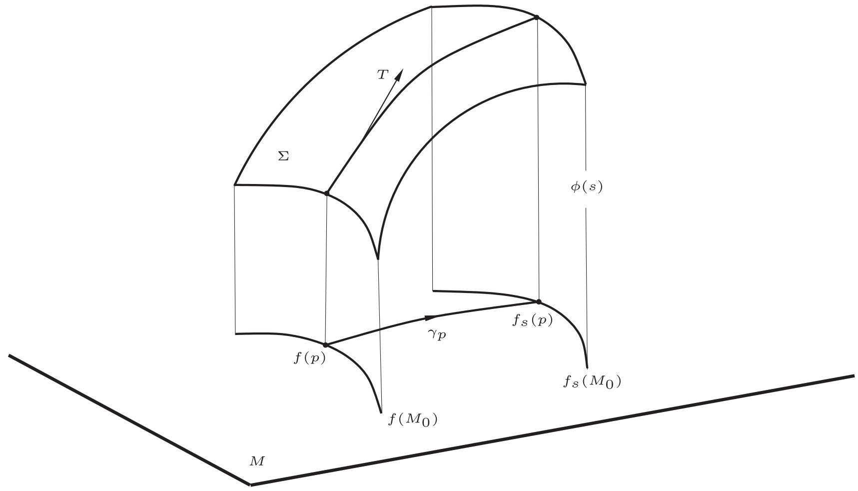

FIGURE 1 An (f_s, ϕ)-graph in $M \times \mathbb{R}$.

3 | GRAPHS ON PARALLEL HYPERSURFACES

Let M_0^{n-1} and M^n be two orientable Riemannian manifolds. Assume that

$$f : M_0^{n-1} \to M^n$$

is an oriented embedding with unit normal field η, and suppose that there is an open interval $I \ni 0$ such that, for all $p \in M_0$, the curve

$$\gamma_p(s) = \exp_M(f(p), s\eta(p)),\ s \in I, \tag{3.1}$$

is a well-defined geodesic of M without conjugate points. In this setting, for any fixed $s \in I$, the map

$$\begin{aligned} f_s :\ \ & M_0 \to M \\ & p \ \mapsto \gamma_p(s) \end{aligned}$$

is an embedding of M_0 into M, which is said to be *parallel* to f. Observe that, given $p \in M_0$, the tangent space $f_{s_*}(T_pM_0)$ of f_s at p is the parallel transport of $f_*(T_pM_0)$ along γ_p from 0 to s. We also remark that, with the induced metric, we will consider the unit normal η_s of f_s at p given by

$$\eta_s(p) = \gamma_p'(s).$$

Having set the notation of the parallel immersions f_s, we introduce now the concept of (f_s, ϕ)-graph, which will play a fundamental role to prove some of the main results of the paper.

Definition 3.1. Let $\phi : I \to \phi(I) \subset \mathbb{R}$ be an increasing diffeomorphism, that is, $\phi' > 0$. With the above notation, we call the set

$$\Sigma := \{(f_s(p), \phi(s)) \in M \times \mathbb{R}\,;\ p \in M_0,\ s \in I\}, \tag{3.2}$$

the *graph* determined by $\{f_s\,;\ s \in I\}$ and ϕ, or (f_s, ϕ)-*graph*, for short (Figure 1).

For an arbitrary point $x = (f_s(p), \phi(s))$ of an (f_s, ϕ)-graph Σ, one has

$$T_x\Sigma = f_{s_*}(T_pM_0) \oplus \text{Span}\{\partial_s\},\ \ \partial_s = \eta_s + \phi'(s)\partial_t.$$

So, a unit normal to Σ is

$$N = \frac{-\phi'}{\sqrt{1+(\phi')^2}}\eta_s + \frac{1}{\sqrt{1+(\phi')^2}}\partial_t\,. \tag{3.3}$$

In particular, its angle function is

$$\Theta = \frac{1}{\sqrt{1+(\phi')^2}}\,\cdot \tag{3.4}$$

As shown in [9, Theorem 6], any (f_s, ϕ)-graph Σ has the *T-property*, meaning that T is a principal direction at any point of Σ. More precisely, one has

$$AT = \frac{\phi''}{(\sqrt{1+(\phi')^2})^3}T. \tag{3.5}$$

Given an (f_s, ϕ)-graph Σ, let $\{X_1, \dots, X_n\}$ be an orthonormal frame of principal directions of Σ in which $X_n = T/\|T\|$. In this case, for $1 \le i \le n-1$, the fields X_i are all horizontal, that is, tangent to M, and constitute principal directions of the immersions f_s at corresponding points (cf. [9, Lemma 1])). Therefore, setting

$$\varrho := \frac{\phi'}{\sqrt{1+(\phi')^2}} \tag{3.6}$$

and considering Equation (3.3), we have, for all $i = 1, \dots, n-1$, that

$$k_i = \langle AX_i, X_i\rangle = -\langle \overline{\nabla}_{X_i} N, X_i\rangle = \varrho\langle \overline{\nabla}_{X_i}\eta_s, X_i\rangle = -\varrho k_i^s,$$

where k_i^s is the ith principal curvature of f_s. Also, it follows from Equation (3.5) that $k_n = \varrho'$. Thus, the principal curvatures of the (f_s, ϕ)-graph Σ at $(f_s(p), \phi(s)) \in \Sigma$ are

$$k_i = -\varrho(s)k_i^s(p)\ (1 \le i \le n-1) \quad \text{and} \quad k_n = \varrho'(s). \tag{3.7}$$

We remark that, up to a constant, the function ϱ defined in Equation (3.6) determines the function ϕ. Indeed, it follows from equality (3.6) that

$$\phi(s) = \int_{s_0}^{s} \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}}du + \phi(s_0), \quad s_0, s \in I. \tag{3.8}$$

It should also be noticed that, from Equations (3.4) and (3.6), the unit normal N defined in Equation (3.3) can be written as $N = -\varrho\eta_s + \Theta\partial_t$. Hence, the relation

$$\varrho^2 + \Theta^2 = 1 \tag{3.9}$$

holds everywhere on any (f_s, ϕ)-graph Σ. In particular, $\varrho = \|T\|$ on Σ.

Definition 3.2. A family $\mathscr{F} := \{f_s : M_0 \to M\,;\ s \in I\}$ of parallel hypersurfaces is called *isoparametric* if, for each $s \in I$, any principal curvature k_i^s of $f_s \in \mathscr{F}$ is constant (possibly depending on i and s). If so, each hypersurface f_s is also called *isoparametric.*

It follows from Equation (3.7) that, if $\mathscr{F} := \{f_s : M_0 \to M\,;\ s \in I\}$ is isoparametric and Σ is an (f_s, ϕ)-graph in $M \times \mathbb{R}$, then all principal curvatures k_i of Σ at any point $(f_s(p), \phi(s))$ are functions of s alone.

4 | ELLIPTIC WEINGARTEN HYPERSURFACES OF $M \times \mathbb{R}$

Let $\Gamma \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ containing the *positive cone*

$$\Gamma_+ := \{\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{R}^n\,;\, k_i > 0\}.$$

A symmetric function $W \in C^\infty(\Gamma)$ will be called a *Weingarten function.* If, in addition, W satisfies the condition:

$$\frac{\partial W}{\partial k_i}(\mathbf{k}) > 0 \quad \forall \mathbf{k} \in \Gamma_+ \text{ and } i = 1, \dots, n, \tag{4.1}$$

then W will be called an *elliptic Weingarten function.*

Example 1. Two distinguished elliptic Weingarten functions are the following:

i) $W(k_1, \dots, k_n) = \sum_{i_1 < \cdots < i_r} k_{i_1} \dots k_{i_r}, \quad r \in \{1, \dots, n\}.$
ii) $W(k_1, \dots, k_n) = \sqrt{k_1^2 + \cdots + k_n^2}.$

The function in (i) is the nonnormalized rth mean curvature H_r (notice that $H_1 = k_1 + \cdots + k_n$ is the *mean curvature* and $H_n = k_1 \dots k_n$ is the *Gauss–Kronecker curvature*), whereas the function in (ii) is the *norm of the second fundamental form* $\|A\|$. We add that these functions are both homogeneous of degree one. (Recall that, given $d \in \mathbb{R}$, a function $f = f(\mathbf{k})$ defined in a cone $\Gamma \subset \mathbb{R}^n$ is said to be *homogeneous of degree d* if $f(t\mathbf{k}) = t^d f(\mathbf{k})$ whenever $\mathbf{k} \in \Gamma$ and $t > 0$.)

Example 2. Given $W = W(k_1, \dots, k_n) \in C^\infty(\Gamma_+)$, define its *inverse* W^* as

$$W^*(k_1, \dots, k_n) := 1/W(1/k_1, \dots, 1/k_n).$$

It is easily checked that W is homogeneous elliptic Weingarten if and only if W^* is homogeneous elliptic Weingarten.

Following [18], given an open set $\Gamma \subset \mathbb{R}^n$ with $\Gamma_+ \subset \Gamma$, we say that

$$W = W(k_1, \dots, k_n, \Theta^2), \quad (k_1, \dots, k_n, \Theta^2) \in \Gamma \times [0, 1],$$

is a *general Weingarten function* (resp. a *general elliptic Weingarten function*) if, for any fixed $\Theta \in [0, 1]$, the map

$$(k_1, \dots, k_n) \in \Gamma \quad \mapsto \quad W(k_1, \dots, k_n, \Theta^2) \in \mathbb{R}$$

is a Weingarten function (resp. an elliptic Weingarten function).

Definition 4.1. We say that a hypersurface Σ of a Riemannian product $M \times \mathbb{R}$ is a *Weingarten hypersurface* if its principal curvatures $k_1, \dots, k_n$, together with its angle function Θ, satisfy a relation of the type

$$W(k_1, \dots, k_n, \Theta^2) = 0, \tag{4.2}$$

where W is a general Weingarten function. More specifically, we shall say that such a Σ is a W-hypersurface. If, in addition, W is elliptic and Σ is strictly convex, then Σ will be called an *elliptic Weingarten hypersurface*.

Hypersurfaces of $M \times \mathbb{R}$ with constant mean curvature H_r are canonical examples of Weingarten hypersurfaces. In the next section, we shall see that hypersurfaces of the CSC in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ are also Weingarten hypersurfaces.

Remark 4.2. We point out that the Hopf Maximum Principle applies to elliptic Weingarten hypersurfaces (see [16] for a detailed discussion in the case $n = 2$). The same is true for the continuation principle, by the results in [21, 24].

The following lemma, which plays a fundamental role here, characterizes Weingarten (f_s, ϕ)-graphs (with $\{f_s\}$ isoparametric) as those whose associated ϱ-functions are solutions of a certain first-order ordinary differential equation. In fact, it follows directly from the definition of Weingarten hypersurface, relations (3.7), and equality (3.9).

Lemma 4.3. *Let Σ be an (f_s, ϕ)-graph in $M \times \mathbb{R}$ whose associated family*

$$\mathscr{F} := \{f_s : M_0 \to M\,;\, s \in I\}$$

of parallel hypersurfaces is isoparametric. Then, given a Weingarten function $W \in C^\infty(\Gamma)$, we have that Σ is a W-hypersurface of $M \times \mathbb{R}$ if and only if its ϱ-function satisfies the equality

$$W(-k_1^s \varrho(s), \dots, -k_{n-1}^s \varrho(s), \varrho'(s), 1 - \varrho^2) = 0, \tag{4.3}$$

where $k_1^s, \dots, k_{n-1}^s$ are the principal curvatures of $f_s \in \mathscr{F}$.

It follows from Lemma 4.3 and Picard's theorem for the local existence of solutions of first-order ODE's that, for any general Weingarten function W, there exist local W-hypersurfaces in $M \times \mathbb{R}$, provided that M admits families of isoparametric hypersurfaces. In this context, it is natural to ask under which conditions we can construct complete W-hypersurfaces in $M \times \mathbb{R}$. We shall pursue this question in the next sections.

5 | ROTATIONAL WEINGARTEN HYPERSURFACES OF $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ AND $\mathbb{H}_\mathbb{F}^m \times \mathbb{R}$

Let M^n be either a simply connected space form $\mathbb{Q}_\epsilon^n$ or a rank-one symmetric space of noncompact type $\mathbb{H}_\mathbb{F}^m$. Take a point $o \in M$ and let $f_s : \mathbb{S}^{n-1} \to M$ be the isometric immersion such that $f_s(\mathbb{S}^{n-1})$ is the geodesic sphere of M with center at o and radius $s > 0$. In this setting, define

$$\mathscr{F} := \{f_s : \mathbb{S}^{n-1} \to M\,;\, s \in (0, \mathcal{R}_M)\}, \tag{5.1}$$

where $\mathcal{R}_M$ is given by

$$\mathcal{R}_M := \begin{cases} +\infty & \text{if} \quad M = \mathbb{R}^n \text{ or } \mathbb{H}_\mathbb{F}^m, \\ \pi/2 & \text{if} \quad M = \mathbb{S}^n. \end{cases} \tag{5.2}$$

As is well known, $\mathscr{F}$ is isoparametric and each sphere $f_s(\mathbb{S}^{n-1})$ is strictly convex (see [12], and references therein). In accordance to the notation of Section 3, for each $s \in (0, +\infty)$, we choose the outward orientation of f_s, so that *any principal curvature k_i^s of f_s is negative*.

In what follows, for M and $\mathscr{F}$ as above, we construct complete strictly convex Weingarten hypersurfaces in $M \times \mathbb{R}$ from (f_s, ϕ)-graphs, $f_s \in \mathscr{F}$. Since the elements of $\mathscr{F}$ are concentric geodesic spheres, we shall call such a hypersurface *rotational*.

The general idea for this construction is to consider Equation (4.3) as an ODE with variable ϱ. From a suitable solution to this equation, we obtain a function ϕ (using Equation (3.8)) which, by Lemma 4.3, defines a Weingarten graph Σ' in $M \times \mathbb{R}$ over an open ball $B_\delta(o) \subset M$ with $\delta \leq +\infty$. If $\delta = +\infty$, Σ' is complete and we are done. Otherwise, $\partial\Sigma'$ is an $(n-1)$-sphere in a horizontal hyperplane $P_t := M \times \{t\}$, and the tangent spaces of Σ' along its boundary are all vertical (i.e., parallel to ∂_t). Hence, a complete Weingarten n-sphere is obtained by "gluing" Σ' with its reflection over P_t along their common boundary.

As we shall see, the effectiveness of this procedure depends on the existence of a solution ϱ to Equation (4.3) which can be defined at the singular point $s = 0$. This fact leads us to introduce the concept of *admissible* Weingarten function, as given below. (Note that the principal curvatures k_i^s of the spheres f_s are not defined at $s = 0$.)

Definition 5.1. Let M and $\mathscr{F}$ be as above. We say that a general Weingarten function W is *M-admissible* (or simply *admissible*) if Equation (4.3) has a solution ϱ defined in $[0, \delta)$, $0 < \delta \leq \mathcal{R}_M$, which satisfies the conditions:

(C1) $\varrho(0) = 0$.
(C2) $0 < \varrho < 1$ on $(0, \delta)$.
(C3) $\varrho' > 0$ on $(0, \delta)$.
(C4) The limits $\lim_{s\to 0} \varrho(s) k_i^s$ and $\lim_{s\to 0} \varrho'(s)$ exist and are finite (recall that k_i^s is the ith principal curvature of f_s) and, if $\delta < \mathcal{R}_M$, the limit $\lim_{s\to\delta} \varrho'(s)$ also exist and is finite .

We assume that δ is maximal with respect to (C2)–(C3) and call $\varrho : [0, \delta) \to [0, 1)$ an *associated function* to W. In the case $\delta < \mathcal{R}_M$, we will write (with a slight abuse of notation):

$$\varrho(\delta) := \lim_{s\to\delta} \varrho(s) \quad \text{and} \quad \varrho'(\delta) := \lim_{s\to\delta} \varrho'(s).$$

(Note that, from the maximality of δ, we must have $\varrho(\delta) = 1$ or $\varrho'(\delta) = 0$.)

Remark 5.2. From Lemma 4.3 and equalities (3.7) and (3.8), a function ϱ satisfying conditions (C2) and (C3) defines a rotational W-graph Σ over the punctured open ball $B_\delta(o) - \{o\} \subset M$, whose function ϕ, up to a constant, is given by

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du, \quad s \in (0, \delta). \tag{5.3}$$

Also, the condition (C1) and the finiteness of the two first limits in (C4) (together with Equation (3.7)) imply that Σ extends C^2-smoothly to the puncture o. Analogously, when $\delta < +\infty$, the finiteness of the last limit in (C4) gives that Σ extends C^2-smoothly to its boundary $f_\delta(\mathbb{S}^{n-1}) \times \{\phi(\delta)\}$.

Now, we are in position to state and prove our first main result.

Theorem 5.3. *Let M^n be either a simply connected space form $\mathbb{Q}_\epsilon^n$ or a rank-one symmetric space of noncompact type $\mathbb{H}_\mathbb{F}^m$. Given an M-admissible Weingarten function W, let $\varrho : (0, \delta) \to [0, 1)$ be its associated function. Then, the following assertions hold:*

i) If $\delta < \mathcal{R}_M$, $\varrho(\delta) = 1$ and $\varrho'(\delta) > 0$, there exists an embedded strictly convex rotational W-sphere in $M \times \mathbb{R}$ which is symmetric with respect to a horizontal hyperplane.
ii) If $\delta = +\infty$ (so that M is $\mathbb{R}^n$ or $\mathbb{H}_\mathbb{F}^m$), there exists a rotational strictly convex entire W-graph in $M \times [0, +\infty)$ which is tangent to $M \times \{0\}$ at a single point, and whose height function is unbounded above.

In particular, if W is elliptic, the W-hypersurfaces in (i) and (ii) are elliptic. Consequently, if (ii) occurs for such a W, there is no compact W-hypersurface in $M \times \mathbb{R}$.

Proof. Assume the hypotheses in (i), and let Σ' be the rotational (f_s, ϕ)-graph defined by the function ϕ in Equation (5.3). As we pointed out in Remark 5.2, Σ' is defined over $B_\delta(o) \subset M$ and constitutes a W-hypersurface of $M \times \mathbb{R}$. Also, by Equation (3.7), Σ' is strictly convex.

Let us show that the function ϕ defining Σ' is bounded in $(0, \delta)$. Indeed, since $\varrho'(\delta) > 0$, there exist $a, \delta_0 > 0$, $0 < \delta - \delta_0 < \delta$, such that $\varrho'(s) \geq a \; \forall s \in (\delta - \delta_0, \delta)$. Besides, $0 \leq \varrho < 1$ and $\varrho(\delta) = 1$. Hence,

$$\begin{aligned}\int_{\delta-\delta_0}^{\delta} \frac{\varrho(s)ds}{\sqrt{1-\varrho^2(s)}} &\leq \int_{\delta-\delta_0}^{\delta} \frac{\varrho'(s)ds}{\varrho'(s)\sqrt{1-\varrho^2(s)}} \leq \frac{1}{a}\int_{\varrho(\delta-\delta_0)}^{1} \frac{d\varrho}{\sqrt{1-\varrho^2}} \\ &= \frac{1}{a}\left(\frac{\pi}{2} - \arcsin(\varrho(\delta-\delta_0))\right) \leq \frac{\pi}{2a},\end{aligned}$$

which implies that ϕ is bounded. So, we can set $\phi(\delta)$ for the limit of $\phi(s)$ as $s \to \delta$.

Since $\varrho(\delta) = 1$, we have from Equation (5.3) that $\phi'(s) \to +\infty$ as $s \to \delta$. This gives that, along $\partial\Sigma' = f_\delta(\mathbb{S}^{n-1}) \times \{\phi(\delta)\}$, the tangent spaces are all parallel to ∂_t. Therefore, since Σ' extends C^2-smoothly to its boundary (see Remark 5.2), if we

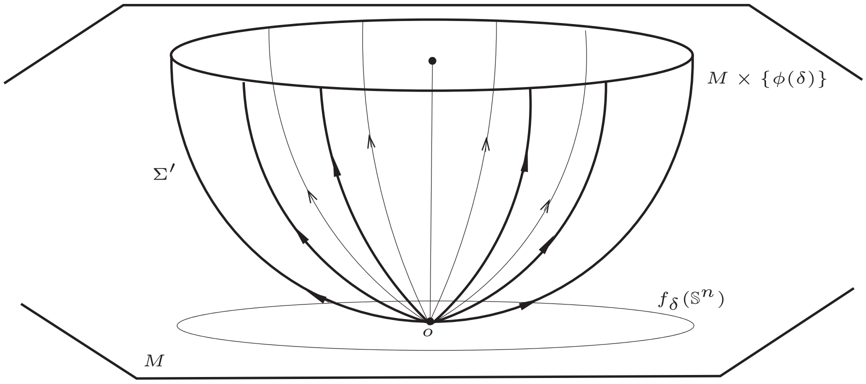

FIGURE 2 A W-hemisphere in $M \times \mathbb{R}$ and its T-trajectories.

TABLE 1 Definition of $\cos_\epsilon$ and $\sin_\epsilon$.

Function	$\epsilon = 0$	$\epsilon = 1$	$\epsilon = -1$
$\cos_\epsilon(s)$	1	$\cos s$	$\cosh s$
$\sin_\epsilon(s)$	s	$\sin s$	$\sinh s$

denote by Σ'' the reflection of Σ' with respect to $M \times \{\phi(\delta)\}$, we have that

$$\Sigma := \text{closure}\, \Sigma' \cup \text{closure}\, \Sigma''$$

is a strictly convex rotational W-sphere of $M \times \mathbb{R}$. This proves (i).

Now, let us assume that the hypotheses in (ii) hold. In this case, since $\delta = +\infty$, the (f_s, ϕ)-graph Σ defined by ϕ in Equation (5.3) is an entire rotational W-graph of $M \times \mathbb{R}$. Also, since $\phi(0) = 0$ and $\phi(s) > 0$ for any $s > 0$, Σ is contained in the closed half-space $M \times [0, +\infty)$, being tangent to $M \times \{0\}$ at o.

It remains to prove that the height function of Σ is unbounded above. For that, we fix $\delta_0 > 0$ and set $\iota_{\delta_0}(s)$ for the infimum of the function $u \mapsto \varrho(u)/\sqrt{1 - \varrho^2(u)}$ on $[\delta_0, s)$, $s > \delta_0$. It is easily seen that $\iota_{\delta_0}(s)$ is bounded away from zero. Therefore,

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du \geq \int_{\delta_0}^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du \geq \iota_{\delta_0}(s)(s - \delta_0) \quad \forall s > \delta_0,$$

which gives that ϕ is unbounded above.

The last assertion regarding the nonexistence of compact elliptic W-hypersurfaces in the occurrence of (ii) follows from the maximum principle (see Remark 4.2) (Figure 2). □

In what follows, for $\epsilon \in \{0, -1, 1\}$, we shall consider the trigonometric functions $\tan_\epsilon = \sin_\epsilon / \cos_\epsilon$ and $\cot_\epsilon = 1/\tan_\epsilon$, where $\cos_\epsilon$ and $\sin_\epsilon$ are defined as in Table 1.

Example 3. Given constants a, b, and c with $a + b \neq 0$, and $\epsilon = \pm 1$, consider the following elliptic general Weingarten function $W_\epsilon : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}$:

$$W_\epsilon(k_1, k_2, \Theta^2) = k_1 k_2 - \frac{c - \epsilon a \Theta^2}{a + b} \cdot$$

For $n = 2$, Gauss equation (2.4) reduces to $K = H_2 + \epsilon\Theta^2$, where K is the intrinsic curvature of Σ and $H_2 = k_1 k_2$ is its extrinsic curvature. Hence, a W_ϵ-surface Σ of $\mathbb{Q}^2_\epsilon \times \mathbb{R}$ satisfies the relation

$$aK + bH_2 = c.$$

(Such surfaces were studied in [16].)

Let us see that W_ϵ is $\mathbb{Q}^2_\epsilon$-admissible, provided that

$$a\epsilon > 0, \quad a+b > 0, \quad \text{and} \quad c - a\epsilon > 0. \tag{5.4}$$

To that end, consider a family $\mathscr{F}$ of concentric geodesic circles in $\mathbb{Q}^2_\epsilon$, and recall that, with the outward orientation, a geodesic circle of radius s in $\mathbb{Q}^2_\epsilon$ has curvature $k(s) = -\cot_\epsilon(s)$. In this setting, the ODE (4.3) takes the form

$$(a+b)\cot_\epsilon(s)\varrho(s)\varrho'(s) + \epsilon a(1-\varrho^2(s)) = c. \tag{5.5}$$

Separating variables and integrating, one easily concludes that the solution ϱ of Equation (5.5) satisfying $\varrho(0) = 0$ is given by

$$\varrho(s) = \left(\frac{c - a\epsilon}{a\epsilon}((\cos_\epsilon(s))^{\frac{-2a}{a+b}} - 1)\right)^{1/2}.$$

In particular, ϱ is increasing and satisfies $\varrho(\delta) = 1$, where

$$\delta := \left(\frac{c}{c - a\epsilon}\right)^{\frac{a+b}{-2a}}.$$

Also, from Equation (5.5), we have that

$$\varrho'(\delta) = \frac{c}{a+b}\tan_\epsilon(\delta) > 0.$$

In addition, a direct application of L'Hâpital's rule gives that

$$\lim_{s\to 0}(\varrho(s)\cot_\epsilon(s)) = \left(\frac{c - a\epsilon}{a+b}\right)^{1/2}.$$

This limit, together with Equation (5.5), yields

$$\lim_{s\to 0}\varrho'(s) = \frac{c - a\epsilon}{\sqrt{(a+b)(c-a\epsilon)}},$$

which completes the proof that W_ϵ is $\mathbb{Q}^2_\epsilon$-admissible with associated function ϱ.

Clearly, ϱ satisfies the conditions of Theorem 5.3-(i). Therefore, for all constants a, b, and c satisfying Equation (5.4), there exists a rotational elliptic W_ϵ-sphere in $\mathbb{Q}^2_\epsilon \times \mathbb{R}$.

Example 4. Given $b < 0 < a$ and an integer $n \geq 2$, write $\alpha := (-a/b)^{\frac{1}{n-1}}$ and set

$$c = a(n-1)\alpha. \tag{5.6}$$

Under these conditions, the Weingarten function $W \in C^\infty(\mathbb{R}^n)$ given by

$$W = aH + bH_n - c,$$

is $\mathbb{Q}^n_\epsilon$-admissible. Indeed, considering Lemma 4.3 for W, $M = \mathbb{Q}^n_\epsilon$, and $\mathscr{F}$ as in Equation (5.1), we have that the ODE (4.3) takes the form

$$a\big((n-1)\cot_\epsilon(s)\varrho(s) + \varrho'(s)\big) + b((\cot_\epsilon(s)\varrho(s))^{n-1}\varrho'(s)) = c. \tag{5.7}$$

Thus, defining

$$\delta := \begin{cases} \arctan_\epsilon(1/\alpha) & \text{if} \quad \epsilon \neq -1 \text{ or } \epsilon = -1 \text{ and } \alpha > 1, \\ +\infty & \text{if} \quad \epsilon = -1 \quad \text{and} \quad \alpha \leq 1, \end{cases}$$

and considering Equation (5.6), we have that

$$\varrho(s) = \alpha \tan_\epsilon(s), \quad s \in [0, \delta),$$

is the solution of Equation (5.7) satisfying $\varrho(0) = 0$. Moreover, for $\delta < +\infty$, we have that $\varrho(s) \to 1$ as $s \to \delta$. It is also clear that ϱ satisfies the conditions (C1)–(C4) of Definition 5.1, which implies that W is $\mathbb{Q}^n_\epsilon$-admissible.

Therefore, by Theorem 5.3, there exists a rotational strictly convex W-hypersurface Σ in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ which is a sphere, if $\delta < +\infty$, or an entire graph, if $\delta = +\infty$. We remark that the principal curvatures $k_1, \dots, k_{n-1}$ of Σ are all constant and equal to α. For $\epsilon = 0$,Σ is the totally geodesic sphere of radius $1/\alpha$.

Let M be either $\mathbb{H}^m_\mathbb{F}$ of $\mathbb{S}^n$. It was proved in [8] that, for all $c > 0$, the elliptic Weingarten function $W_c = H_r - c$ is M-admissible. Moreover, for $M = \mathbb{H}^m_\mathbb{F}$, there is a constant $C(\mathbb{F}) > 0$ such that the associated function ϱ_c to W_c satisfies the conditions of Theorem 5.3-(i) (resp. Theorem 5.3-(ii)) if $c > C(\mathbb{F})$ (resp. $c \leq C(\mathbb{F})$). In our next result, we show that this situation is somewhat typical.

Theorem 5.4. *Let M be as in Theorem 5.3. Given $c > 0$, let W_c be the Weingarten function defined by $W_c = f - c$, where $f = f(k_1, \dots, k_n)$ is a symmetric homogeneous function defined on an open cone $\Gamma \supset \Gamma_+$ in $\mathbb{R}^n$. Suppose that, for some $c_0 > 0$, W_{c_0} is M-admissible with associated function $\varrho_0 : [0, \delta_0) \to [0, 1)$. Then, one has:*

i) If $\delta_0 < \mathcal{R}_M$, there is a constant $c_1 \geq c_0$ with the following property: for all $c > c_1$, W_c is M-admissible and there exists an embedded W_c-sphere in $M \times \mathbb{R}$ as in the statement of Theorem 5.3-(i).
ii) If $\delta_0 = +\infty$, for all positive $c \leq c_0$, W_c is M-admissible and there exists an entire rotational W_c-graph in $M \times [0, +\infty)$ as in the statement of Theorem 5.3-(ii).

Proof. First, we present a general construction that will be used along the proof. From the hypothesis, we have that the function $\varrho_0 : [0, \delta_0) \to [0, 1)$ satisfies

$$f(-k_1^s \varrho_0, \dots, -k_{n-1}^s \varrho_0, \varrho_0') = c_0 \,. \tag{5.8}$$

Given $c > 0$, multiplying both sides of Equation (5.8) by c/c_0, and denoting by d the degree of homogeneity of the Weingarten function f, one gets

$$f(-(c/c_0)^{1/d} k_1^s \varrho_0, \dots, -(c/c_0)^{1/d} k_{n-1}^s \varrho_0, (c/c_0)^{1/d} \varrho_0') = c \,,$$

which implies that the function $\overline{\varrho}_c : [0, \delta_0) \to [0, 1)$ defined by

$$\overline{\varrho}_c(s) = \left(\frac{c}{c_0}\right)^{1/d} \varrho_0(s) \tag{5.9}$$

satisfies $W_c(-k_1^s \overline{\varrho}_c(s), \dots, -k_{n-1}^2 \overline{\varrho}_c(s), \overline{\varrho}_c'(s)) = 0$. In particular, we may possibly extend $\overline{\varrho}_c$ past δ_0 or restrict $\overline{\varrho}_c$ to a subinterval to obtain a solution $\varrho_c : [0, \delta_c) \to [0, 1)$ to

$$f(-k_1^s \varrho_c(s), \dots, -k_{n-1}^s \varrho_c(s), \varrho_c'(s)) = c, \tag{5.10}$$

where δ_c is maximal with respect to conditions (C2) and (C3) of Definition 5.1. Concerning the M-admissibility of W_c, it is straightforward to see that ϱ_c satisfies (C1) and the first two conditions of (C4), hence W_c will be M-admissible, if $\delta_c = \infty$ or if, when $\delta_c < \infty$, $\lim_{s \to \delta_c} \varrho_c'(s)$ exists and is finite.

Having defined the family $\{\varrho_c\}_{c>0}$, we next prove (i), so we assume that $\delta_0 < \mathcal{R}_M$. In this case, set

$$c_1 := \frac{c_0}{(\varrho_0(\delta_0))^d} \geq c_0.$$

For a given $c > c_1$, let ϱ_c be defined as above. We claim that $\delta_c < \delta_0$. In fact, if $\delta_c \geq \delta_0$, then the fact that $\varrho_c|_{[0,\delta_0)} = \left(\frac{c}{c_0}\right)^{1/d} \varrho_0$, implies that

$$\varrho_c(\delta_0) = \left(\frac{c}{c_0}\right)^{1/d} \varrho_0(\delta_0) > \left(\frac{c_1}{c_0}\right)^{1/d} \varrho_0(\delta_0) = 1,$$

contradicting the maximality of δ_c with respect to condition (C2) of Definition 5.1. In particular, $\varrho_0'(\delta_c) > 0$ and we have that

$$\lim_{s\to\delta_c} \varrho_{c_1}'(s) = \left(\frac{c}{c_0}\right)^{1/d} \varrho_0'(\delta_c) > 0.$$

This proves that W_c is M-admissible and that $\varrho_c(\delta_c) = 1$. In particular, Theorem 5.3-(i) applies to W_c, and this proves (i).

Let us assume now that $\delta_0 = +\infty$ to prove (ii). Then, for all positive $c \leq c_0$, one has

$$\varrho_c(s) = \left(\frac{c}{c_0}\right)^{1/d} \varrho_0(s) \leq \varrho_0(s) < 1 \ \ \forall s \in [0, +\infty),$$

which clearly implies that $\varrho_c : [0, +\infty) \to [0, 1)$ is well defined and satisfies the conditions (C1)–(C4) of Definition 5.1, showing that W_c is M-admissible. Therefore, Theorem 5.3-(ii) applies. This shows (ii) and finishes our proof. □

Remark 5.5. Theorem 5.4 can be improved under the additional assumption that W_c is uniformly elliptic, in the sense that there exists a constant $a > 0$ such that $\frac{\partial f}{\partial x_i} \geq a$ in Γ^+. In this case, *W_c is M-admissible for any $c > 0$*. To see this, we make use of the construction of $\varrho_c : [0, \delta_c) \to [0, 1)$ as above. As already explained, to prove that W_c is M-admissible it suffices to show that, if $\delta_c < \infty$, $\lim_{s\to\delta_c} \varrho_c'(s)$ exists and is finite.

First, note that if $\delta_c \leq \delta_0$, then $\lim_{s\to\delta_c} \varrho_c'(s) = (c/c_0)^{1/d} \varrho_0'(\delta_c) < \infty$, so we may assume that $\delta_c > \delta_0$. In this case, set

$$\lambda_i = -k_i^{\delta_c} \varrho_c(\delta_c), \ \ i\pi = 1, 2, \dots, n-1.$$

Then, the assumption that $\frac{\partial f}{\partial x_n} \geq a > 0$ implies that $\lim_{x\to\infty} f(\lambda_1, \dots, \lambda_{n-1}, x) = \infty$, hence Equation (5.10) implies that the function $\varrho_c'(s)$ is uniformly bounded.

Next, we prove that the limit $\lim_{s\to\delta_c} \varrho_c'(s)$ exists. Let $(s_m)_{m\in\mathbb{N}}$ and $(t_m)_{m\in\mathbb{N}}$ be two sequences in $(0, \delta_c)$ with $s_m, t_m \nearrow \delta_c$ and such that

$$\lim_{m\to\infty} \varrho_c'(s_m) = \alpha_1 \in [0, \infty), \qquad \lim_{m\to\infty} \varrho_c'(t_m) = \alpha_2 \in [0, \infty).$$

Then, Equation (5.10) implies that

$$f(\lambda_1, \dots, \lambda_{n-1}, \alpha_1) = c = f(\lambda_1, \dots, \lambda_{n-1}, \alpha_2),$$

from where we obtain that $\alpha_1 = \alpha_2$, since $\frac{\partial f}{\partial x_n} > a > 0$. Thus, $\lim_{s\to\delta_c} \varrho'(s)$ exists and is finite, proving that W_c is M-admissible.

Example 5. Given constants $a, b > 0$, let $f : \mathbb{R}^n \to \mathbb{R}$ be the symmetric homogeneous function of degree 2 defined by

$$f = a\|A\|^2 + bH_2.$$

For any given $c > 0$, $W_c = f - c$ is clearly an elliptic Weingarten function. Considering Lemma 4.3 for $M = \mathbb{Q}^n_\epsilon$, $W = W_c$, and $\mathscr{F}$ as in (5.1), the ODE (4.3) takes the form:

$$a(\varrho'(s))^2 + (n-1)b\cot_\epsilon(s)\varrho(s)\varrho'(s) + (\alpha\cot_\epsilon(s)\varrho^2(s) - c) = 0, \tag{5.11}$$

where $\alpha := (n-1)a + b\binom{n-1}{2}$.

Note that, for any $a > 0$, we can choose $b > 0$ in such a way that

$$(n-1)^2b^2 - 4a\alpha = 0. \tag{5.12}$$

Indeed, this equality is equivalent to the quadratic equation for b:

$$(n-1)b^2 - 2(n-2)ab - 4a^2 = 0, \tag{5.13}$$

which is easily seem to have a positive root.

Assuming Equation (5.12), we can solve Equation (5.11) for ϱ', obtaining

$$\varrho'(s) = -\frac{(n-1)b}{2a}\cot_\epsilon(s)\varrho(s) + \frac{\sqrt{ac}}{a}\,. \tag{5.14}$$

Setting $\beta := \frac{(n-1)b}{2a}$ and $\mathcal{R}_\epsilon := \mathcal{R}_{\mathbb{Q}^n_\epsilon}$, as in Equation (5.2), the standard method of resolution of linear ODE's gives that

$$\varrho(s) = \frac{\sqrt{ac}}{a}\frac{\int_0^s \sin_\epsilon^\beta(u)du}{\sin_\epsilon^\beta(s)}, \quad s \in (0, \mathcal{R}_\epsilon),$$

is a (positive) solution to Equation (5.14). A direct computation yields

$$\lim_{s\to 0}\varrho(s) = 0 \quad \text{and} \quad \lim_{s\to 0}(\cot_\epsilon(s)\varrho(s)) = \frac{\sqrt{ac}}{a(\beta+1)}\,. \tag{5.15}$$

Also, from Equation (5.14) and the second equality in Equation (5.15), we have

$$\lim_{s\to 0}\varrho'(s) = \frac{\sqrt{ac}}{a(\beta+1)} > 0. \tag{5.16}$$

Hence, $\varrho'(s) > 0$ for $s > 0$ sufficiently small. In fact, one has $\varrho' > 0$ in $(0, \mathcal{R}_\epsilon)$. Otherwise, there would exist $s_0 > 0$ such that $\varrho'(s_0) = 0$ and $\varrho'(s) > 0$ for all $s \in (0, s_0)$. Then, Equation (5.14) would give

$$\varrho''(s_0) = \frac{\beta\varrho(s_0)}{\sin_\epsilon^2(s_0)} > 0,$$

that is, s_0 would be a local minimum for ϱ, contradicting that $\varrho' > 0$ in $s \in (0, s_0)$.

Again by a direct computation, we have

$$\lim_{s\to\mathcal{R}_\epsilon}\varrho(s) = \begin{cases} +\infty & \text{if} \quad \epsilon = 0, \\ \frac{\sqrt{ac}}{\beta} & \text{if} \quad \epsilon = -1, \\ \sqrt{ac}\,I(\beta) & \text{if} \quad \epsilon = 1, \end{cases}$$

where $I(\beta) := \int_0^{\pi/2} \sin^{\beta}(s)ds \le 1$. Thus, we can define $\delta_c := \varrho^{-1}(1) < \mathcal{R}_\epsilon$ in any of the following occurrences:

- $\epsilon = 0$.
- $\epsilon = -1$ and $\frac{\sqrt{ac}}{\beta} > 1$.
- $\epsilon = 1$ and $\sqrt{ac}\, I(\beta) > 1$.

In any of these cases, it follows from Equation (5.14) that

$$\lim_{s \to \delta_c} \varrho'(s) < +\infty.$$

Finally, we set $\delta_c = +\infty$ if $\epsilon = -1$ and $\frac{\sqrt{ac}}{\beta} \le 1$.

It follows from the above considerations that, for any $a > 0$ and any $b > 0$ satisfying Equation (5.13), $W_c = f - c$ is $\mathbb{Q}_\epsilon^n$-admissible for all $c > 0$. (However, for $\epsilon = 1$, $\delta_c = \pi/2 = \mathcal{R}_\epsilon$ if $\sqrt{ac}\, I(\beta) \le 1$.) Furthermore, Theorem 5.3-(i) applies in the case $\delta_c < \mathcal{R}_\epsilon$, and Theorem 5.3-(ii) applies in the case $\epsilon = -1$ and $\delta_c = +\infty$. (Notice that, for $\epsilon = 0$, the W_c-sphere obtained from Theorem 5.3-(i) is totally umbilical and has radius $R = a(1+\beta)/\sqrt{ac}$.)

6 | SYMMETRIC HYPERSURFACES OF CONSTANT SCALAR CURVATURE IN $\mathbb{Q}_\epsilon^n \times \mathbb{R}$

In this section, we consider hypersurfaces of CSC of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$, $\epsilon \neq 0$, which are *symmetric*, meaning that they are invariant by elliptic, parabolic or hyperbolic isometries. Any such isometry is determined by a parallel family $\mathscr{F}$ of totally umbilical hypersurfaces of $\mathbb{Q}_\epsilon^n$, that is, geodesic spheres (elliptic), horospheres (parabolic) or equidistant hypersurfaces (hyperbolic). In particular, hypersurfaces invariant by elliptic isometries are the rotational ones.

Remark 6.1. Regarding the notation $\mathbb{Q}_\epsilon^n$, we will assume from now on that $\epsilon \neq 0$, that is, $\mathbb{Q}_\epsilon^n$ will refer only to $\mathbb{H}^n$ or $\mathbb{S}^n$.

Let us start by showing that CSC hypersurfaces in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ are Weingarten hypersurfaces. Indeed, assuming $n \ge 3$, consider a hypersurface Σ of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ and set $K(X,Y)$ for its sectional curvature determined by $X, Y \in T\Sigma$. Given an orthonormal frame $\{X_1, \dots, X_n\}$ of principal directions in $T\Sigma$, it follows from Gauss equation (2.3) that:

$$K(X_i, X_j) = k_i k_j + \epsilon(1 - \|T_{ij}\|^2), \quad i \neq j \in \{1, \dots, n\}, \tag{6.1}$$

where k_i is the principal curvature in the direction X_i, and T_{ij} is the orthogonal projection of T on the plane of $T\Sigma$ determined by X_i and X_j.

In this setting, we have that the (nonnormalized) *scalar curvature* S of Σ is:

$$S = \sum_{i \neq j} K(X_i, X_j).$$

Considering the equality $T = \partial_t - \Theta N$ and noticing that

$$\sum_{i \neq j} \|T_{ij}\|^2 = 2(n-1)\|T\|^2 = 2(n-1)(1 - \Theta^2),$$

we have from Equation (6.1) that

$$S = 2H_2 + \epsilon(n-1)(2\Theta^2 + n - 2). \tag{6.2}$$

Therefore, given $c \in \mathbb{R}$, defining $W_c : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ by

$$W_c(k_1, \dots, k_n, \Theta^2) = 2H_2(k_1, \dots, k_n) + \epsilon(n-1)(2\Theta^2 + n - 2) - c, \tag{6.3}$$

we have that W_c is elliptic Weingarten, and also that a W_c-hypersurface of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ has CSC $S = c$.

Now, choose a family

$$\mathscr{F} := \{f_s : M_0 \to \mathbb{Q}_\epsilon^n\,;\, s \in I\}$$

of parallel totally umbilical hypersurfaces of $\mathbb{Q}_\epsilon^n$, and write $\alpha(s)$ for the principal curvature of $f_s \in \mathscr{F}$. In this setting, the equalities (3.7) take the form:

$$k_i = -\alpha\varrho \ \ (i = 1, \dots, n-1) \quad \text{and} \quad k_n = \varrho',$$

which gives

$$2H_2(k_1, \dots, k_n) = -2(n-1)\alpha\varrho\varrho' + (n-1)(n-2)\varrho^2\alpha^2.$$

Since $\Theta^2 = 1 - \varrho^2$, we have that the equality $W_c(k_1, \dots, k_n) = 0$ for $k_1, \dots, k_n$ as above (that is, the ODE (4.3) for $W = W_c$) is equivalent to

$$(n-1)(-2\alpha\varrho\varrho' + ((n-2)\alpha^2 - 2\epsilon)\varrho^2 + n\epsilon) = c. \tag{6.4}$$

Setting $\tau := \varrho^2$, Equation (6.4) becomes

$$(n-1)(-\alpha\tau' + ((n-2)\alpha^2 - 2\epsilon)\tau + n\epsilon) = c. \tag{6.5}$$

Equation (6.1) also gives that the sectional curvatures $K(X_i, X_j)$ of an (f_s, ϕ)-graph Σ as in Lemma 4.3 are given by (recall that T is a principal direction of Σ):

$$\begin{aligned} K(X_i, X_j) &= \alpha^2\varrho^2 + \epsilon \quad (i \neq j = 1, \dots, n-1). \\ K(X_i, X_n) &= -\alpha\varrho\varrho' + \epsilon(1 - \varrho^2) \quad (i = 1, \dots, n-1). \end{aligned} \tag{6.6}$$

In particular, Σ has constant sectional curvature if and only if ϱ satisfies

$$\alpha\varrho\varrho' + (\alpha^2 + \epsilon)\varrho^2 = 0. \tag{6.7}$$

Remark 6.2. Throughout this section, W_c will always denote the Weingarten function defined in Equation (6.3). We stress that $\Sigma \subset \mathbb{Q}_\epsilon^n \times \mathbb{R}$ has the CSC c if and only if Σ is a W_c-hypersurface.

6.1 | Rotational constant scalar curvature hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$

Setting $\mathcal{R}_\epsilon := \mathcal{R}_{\mathbb{Q}_\epsilon^n}$, as in Equation (5.2), we have that any principal curvature of a geodesic sphere of $\mathbb{Q}_\epsilon^n$ of radius $s \in (0, \mathcal{R}_\epsilon)$ is $\alpha(s) = -\cot_\epsilon s$. In this case, Equation (6.5) takes the form

$$\tau'(s) = a(s)\tau(s) + b(s), \quad s \in (0, \mathcal{R}_\epsilon), \tag{6.8}$$

where the functions a and b are given by

$$a = -(n-2)\cot_\epsilon + 2\epsilon\tan_\epsilon \qquad \text{and} \qquad b = \left(\frac{c}{n-1} - \epsilon n\right)\tan_\epsilon. \tag{6.9}$$

The general solution to Equation (6.8) is as follows. For fixed $s_0 \in (0, +\infty)$ and $\tau_0 \in \mathbb{R}$,

$$\tau(s) = \frac{1}{\mu(s)}\left(\tau_0 + \int_{s_0}^{s} b(u)\mu(u)du\right), \tag{6.10}$$

where $\mu(s) = \exp\left(-\int_{s_0}^{s} a(u)du\right)$ and $s \in (0, +\infty)$. A direct computation gives

$$\tau(s) = \mathfrak{C}_n \frac{\sin_\epsilon^n(s) - \sin_\epsilon^n(s_0)}{\sin_\epsilon^{n-2}(s)\cos_\epsilon^2(s)} + \tau_0 \frac{\sin_\epsilon^{n-2}(s_0)\cos_\epsilon^2(s_0)}{\sin_\epsilon^{n-2}(s)\cos_\epsilon^2(s)}, \tag{6.11}$$

where $\mathfrak{C}_n$ is the constant defined as

$$\mathfrak{C}_n := \frac{c - \epsilon n(n-1)}{n(n-1)} . \tag{6.12}$$

Note that there exists a solution τ which is defined at the singular point $s_0 = 0$ and satisfies $\tau(0) = 0$. Namely,

$$\tau(s) = \mathfrak{C}_n \tan_\epsilon^2(s). \tag{6.13}$$

In this case, since τ is required to be a nonnegative function, we must have $\mathfrak{C}_n \geq 0$, that is $c \geq \epsilon n(n-1)$. If $c = \epsilon n(n-1)$, the function τ, and so ϱ, vanishes identically, and the corresponding (f_s, ϕ)-graph is nothing but a totally geodesic horizontal hyperplane of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$. So, we shall assume $\mathfrak{C}_n > 0$.

If $c > 0$ and $\epsilon = -1$, we have that $\mathfrak{C}_n > 1$, so that $\arctan_\epsilon(1/\sqrt{\mathfrak{C}_n})$ is well defined for these values of c and ϵ (as well as for $c > 0$ and $\epsilon = 1$). On the other hand, $\mathfrak{C}_n \leq 1$ if $c \leq 0$ and $\epsilon = -1$. These facts and equality (6.13) imply that, in any case, W_c is $\mathbb{Q}_\epsilon^n$-admissible and its associated function ϱ is

$$\varrho(s) = \sqrt{\mathfrak{C}_n}\tan_\epsilon(s), \ \ s \in [0, \delta), \tag{6.14}$$

where $\delta > 0$ is given by

$$\delta := \begin{cases} \arctan_\epsilon(1/\sqrt{\mathfrak{C}_n}) & \text{if} \quad c > 0 \\ +\infty & \text{if} \quad c \leq 0. \end{cases}$$

(Note that the condition (C4) in Definition 5.1 is easily checked.) In particular, ϱ fulfills the conditions of Theorem 5.3, where case (i) occurs if $c > 0$, and case (ii) occurs if $c \leq 0$. Finally, a direct computation gives that ϱ satisfies Equation (6.7).

Summarizing, we have the following result.

Theorem 6.3. *Given $n \geq 3$, for all $c > \epsilon n(n-1)$, there exists a properly embedded strictly convex (and so elliptic) rotational W_c-hypersurface Σ in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ which is necessarily of constant sectional curvature $K = c/(n(n-1)) > \epsilon$. Furthermore, if $c > 0$, Σ is a sphere as in Theorem 5.3-(i), and if $c \leq 0$, Σ is an entire graph as in Theorem 5.3-(ii).*

Let us consider Equation (6.8) again and look for solutions satisfying $\tau(\lambda) = 1$ and $\tau'(\lambda) < 0$ for suitable values of $s_0 = \lambda$. In this case, we must have $a(\lambda) + b(\lambda) < 0$, which yields the inequality

$$(n\mathfrak{C}_n + 2\epsilon)\tan_\epsilon^2(\lambda) < (n-2). \tag{6.15}$$

Assume $c > \epsilon n(n-1)$ and define $\delta_\epsilon(c) \in (0, \infty]$ as

$$\delta_\epsilon(c) = \sup\{\lambda > 0 \mid (n\mathfrak{C}_n + 2\epsilon)\tan_\epsilon^2(\lambda) < (n-2)\}.$$

Then, we have:

Theorem 6.4. *Given $c > \epsilon n(n-1)$, there exists a one-parameter family*

$$\mathcal{S} = \{\Sigma(\lambda) \,;\, \lambda \in (0, \delta_\epsilon(c))\}$$

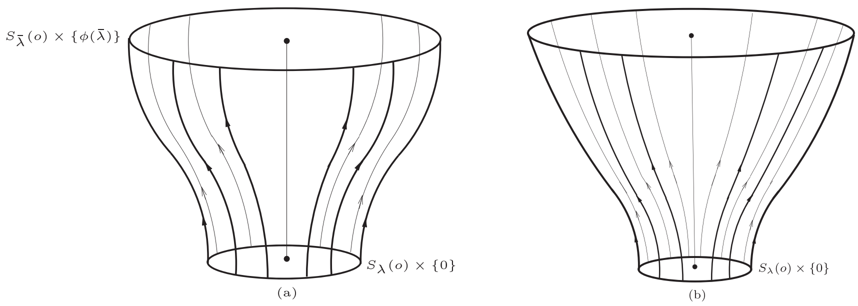

FIGURE 3 The two types of rotational W_c-annuli of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, where the one on the right occurs only for $\epsilon = -1$.

of properly embedded rotational W_c-hypersurfaces in $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ which are all homeomorphic to the n-annulus $\mathbb{S}^{n-1} \times \mathbb{R}$. In addition, the following assertions hold:

i) If either $\epsilon = -1$ and $\mathfrak{C}_n > 1$, or $\epsilon = 1$, each $\Sigma(\lambda) \in \mathcal{S}$ is Delaunay-type, that is, it is periodic in the vertical direction, and has unduloids as its T-trajectories.
ii) If $\epsilon = -1$ and $\mathfrak{C}_n \le 1$, each hypersurface $\Sigma(\lambda) \in \mathcal{S}$ is symmetric with respect to $\mathbb{Q}^n_\epsilon \times \{0\}$ and has unbounded height function.

Proof. Given $s_0 = \lambda \in (0, \delta_\epsilon(c))$, let τ be the solution of Equation (6.8) satisfying $\tau(\lambda) = 1$. Then, from the definition of $\delta_\epsilon(c)$, and equalities (6.8) and (6.9), it follows that $\tau'(\lambda) < 0$, so that τ is decreasing near λ. Since we are assuming $c > \epsilon n(n-1)$, we have that $\mathfrak{C}_n$ is positive, and so is $b = n\mathfrak{C}_n \tan_\epsilon$. This, together with Equation (6.10), implies that τ is positive on $(\lambda, \mathcal{R}_\epsilon)$. Also, from Equation (6.13), one has

$$\lim_{s\to\mathcal{R}_\epsilon} \tau(s) = \begin{cases} \mathfrak{C}_n & \text{if} \quad \epsilon = -1 \\ +\infty & \text{if} \quad \epsilon = 1. \end{cases} \tag{6.16}$$

Therefore, if $\epsilon = 1$ or if $\epsilon = -1$ and $\mathfrak{C}_n > 1$, there exists $\bar\lambda > \lambda$ such that

$$0 < \tau|_{(\lambda,\bar\lambda)} < 1 \quad \text{and} \quad \tau(\lambda) = \tau(\bar\lambda) = 1.$$

Let us see that $\tau'(\bar\lambda) \neq 0$. Assuming otherwise, we have $a(\bar\lambda) = -b(\bar\lambda)$, so that $(n\mathfrak{C}_n + 2\epsilon)\tan^2_\epsilon(\bar\lambda) = n-2$. In particular, $n\mathfrak{C}_n + 2\epsilon > 0$. However,

$$\tau''(\bar\lambda) = a'(\bar\lambda) + b'(\bar\lambda) = \frac{n-2}{\sin^2_\epsilon(\bar\lambda)} + \frac{n\mathfrak{C}_n + 2\epsilon}{\cos^2_\epsilon(\bar\lambda)} > 0,$$

which implies that $\bar\lambda$ is a local minimum of τ. This is a contradiction, since $\bar\lambda$ is a maximum for τ in $(\lambda, \bar\lambda]$. Thus, $\tau'(\bar\lambda) > 0$.

It follows from the above considerations that, setting $\tau_\lambda := \tau|_{(\lambda,\bar\lambda)}$, we can proceed as in the proof of Theorem 5.3 and conclude that the (f_s, ϕ)-graph $\Sigma'(\lambda)$ with ϱ-function $\varrho = \sqrt{\tau_\lambda}$ is a bounded W_c-hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Moreover, $\Sigma'(\lambda)$ is homeomorphic to $\mathbb{S}^{n-1} \times (\lambda, \bar\lambda)$ and has boundary

$$\partial\Sigma'(\lambda) = (S_\lambda(o) \times \{0\}) \cup (S_{\bar\lambda}(o) \times \{\phi(\bar\lambda)\}),$$

where $S_s(o)$ denotes the sphere of $\mathbb{Q}^n_\epsilon$ with radius s and center at o (Figure 3a).

Since $\varrho(\lambda) = \varrho(\bar\lambda) = 1$, the tangent spaces of $\Sigma'(\lambda)$ are vertical along its boundary $\partial\Sigma'(\lambda)$. Moreover, $\Sigma'(\lambda)$ extends C^2-smoothly to $\partial\Sigma'(\lambda)$, for $\tau'(\lambda)$ and $\tau'(\bar\lambda)$ (and so $\varrho'(\lambda)$ and $\varrho'(\bar\lambda)$) are both finite. Therefore, we obtain a properly embedded rotational W_c-hypersurface $\Sigma(\lambda)$ from $\Sigma'(\lambda)$ by continuously reflecting it with respect to the horizontal hyperplanes $\mathbb{Q}^n_\epsilon \times \{k\phi(\bar\lambda)\}$, $k \in \mathbb{Z}$. This proves (i).

To prove (ii), let us suppose that $\epsilon = -1$ and $\mathfrak{C}_n \le 1$. In this case, Equation (6.16) yields

$$0 < \tau|_{(\lambda,+\infty)} < 1,$$

so that the (f_s, ϕ)-graph $\Sigma'(\lambda)$ determined by $\varrho = \sqrt{\tau|_{(\lambda,+\infty)}}$ is a W_c-hypersurface of $\mathbb{H}^n \times \mathbb{R}$ with boundary $\partial\Sigma'(\lambda) = S_\lambda(o) \times \{0\}$ (Figure 3b). By reflecting $\Sigma'(\lambda)$ with respect to $\mathbb{Q}^n_\epsilon \times \{0\}$, as we did before, we obtain the embedded W_c-hypersurface $\Sigma(\lambda)$ as stated.

It remains to show that the height function of $\Sigma(\lambda)$ is unbounded. For that, we have just to observe that the infimum of τ in $[\lambda, +\infty)$ is positive, since τ itself is positive in this interval, and its limit as $s \to +\infty$ is $\mathfrak{C}_n$. So, the same is true for $\varrho = \sqrt{\tau}$. Therefore,

$$\phi(s) = \int_\lambda^s \frac{\varrho(u)}{\sqrt{1-\varrho^2(u)}}du > \int_\lambda^s \varrho(u)du > \inf \varrho|_{[\lambda,+\infty)}(s-\lambda),$$

from which we conclude that ϕ is unbounded. This finishes the proof of (ii). □

6.2 | Translational constant scalar curvature hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$

Now, we consider hypersurfaces of constant negative scalar curvature in $\mathbb{H}^n \times \mathbb{R}$ which are invariant by either parabolic or hyperbolic translations. Recall that, in hyperbolic space $\mathbb{H}^n$, parabolic translations are isometries which fix parallel families of horospheres, whereas hyperbolic translations are those which fix parallel families of equidistant hypersurfaces to a totally geodesic hyperplane of $\mathbb{H}^n$. In the upper half-space model of $\mathbb{H}^n$, horizontal Euclidean translations along fixed horizontal directions are parabolic, and Euclidean homotheties from the origin are hyperbolic. It is easily seen that any of these isometries extend to an isometry of $\mathbb{H}^n \times \mathbb{R}$ which fixes the factor $\mathbb{R}$ pointwise.

Starting with the parabolic case, let us first consider (f_s, ϕ)-graphs such that

$$\mathscr{F} := \{f_s : \mathbb{R}^{n-1} \to \mathbb{H}^n\,;\, s \in (-\infty, +\infty)\}$$

is a parallel family of horospheres of $\mathbb{H}^n$. Since the principal curvatures of any horosphere $\mathscr{H}_s := f_s(\mathbb{R}^{n-1})$ are all equal to 1, Equation (6.5) becomes

$$\tau'(s) = n\tau(s) - \frac{c + n(n-1)}{n-1}\,. \tag{6.17}$$

Consider a constant $c \in [-n(n-1), 0)$ and write

$$b_c = -\frac{c + n(n-1)}{n-1}, \tag{6.18}$$

so that $0 \le -b_c/n < 1$. In this setting, the constant function

$$\tau_c(s) = -\frac{b_c}{n}\,,\ \ s \in (-\infty, +\infty),$$

is a trivial solution of Equation (6.17) satisfying $0 \le \tau_c < 1$. The function $\varrho_c = \sqrt{\tau_c}$ is also constant, and so it is a solution of Equation (6.7) (for $\alpha = 1$ and $\epsilon = -1$). Hence, defining

$$\phi_c(s) = \int_0^s \frac{\varrho_c(u)}{\sqrt{1-\varrho_c^2(u)}}du = \frac{\varrho_c}{\sqrt{1-\varrho_c^2}}s,\ \ s \in (-\infty, +\infty),$$

we have from Lemma 4.3 that the (f_s, ϕ_c)-graph $\Sigma_1(c)$ is entire and has the constant sectional curvature $K = \varrho_c^2 - 1 = -\Theta$. In particular, $\Sigma_1(c)$ has constant angle function, and CSC $c = -n(n-1)\Theta$. Also, from identities (3.7), all principal curvatures of $\Sigma_1(c)$ are nonpositive, so that $\Sigma_1(c)$ is convex (but not strictly convex, since $k_n = \varrho_c' = 0$).

We add that, for $\varrho_c = 0, \Sigma_1(c)$ is a horizontal hyperplane of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ of CSC $c = -n(n-1)$. Moreover, $\tau_c = -b_c/n \to 1$ as $c \to 0$, which implies that the angle function of $\Sigma_1(c)$ goes to 0 as $c \to 0$. Consequently, as $c \to 0, \Sigma_1(c)$ converges uniformly (on compact sets) to the cylinder $\mathscr{H}_0 \times \mathbb{R}$ over the horosphere $\mathscr{H}_0 = f_0(\mathbb{R}^{n-1}) \subset \mathbb{H}^n$. (Note that, for all $c \in [-n(n-1), 0)$, $\Sigma_1(c) \cap (\mathbb{H}^n \times \{0\}) = \mathscr{H}_0$.)

A direct computation also gives that the nonconstant function

$$\tau_c(s) = \left(1 + \frac{b_c}{n}\right)\mathrm{e}^{ns} - \frac{b_c}{n}, \quad s \in (-\infty, 0], \tag{6.19}$$

with b_c as in Equation (6.18), is the solution of Equation (6.17) which satisfies:

$$0 < \tau_c(s) \leq 1 = \tau_c(0) \ \ \forall s \in (-\infty, 0].$$

Moreover, since $\tau_c'(0) > 0$, as in the proof of Theorem 5.3, we have that

$$\phi_c(s) = \int_0^s \frac{\varrho_c(u)}{\sqrt{1 - \varrho_c^2(u)}} du, \quad s \in (-\infty, 0],$$

is well defined.

Assume $c > -n(n-1)$. Considering equality (6.19), one has

$$\varrho_c := \sqrt{\tau} \geq \sqrt{-b_c/n} > 0.$$

Thus, for all $s \in (-\infty, 0)$,

$$-\phi_c(s) = \int_s^0 \frac{\varrho_c(u)}{\sqrt{1 - \varrho_c^2(u)}} du \geq \int_s^0 \varrho_c(u) du \geq \sqrt{-b_c/n}(-s),$$

which implies that ϕ_c is unbounded below. On the other hand, if $c = -n(n-1)$, then $b_c = 0$, which yields $\varrho_c(s) = \mathrm{e}^{ns/2}$. Hence, in this case,

$$-\phi_c(s) = \int_s^0 \frac{\mathrm{e}^{nu/2}}{\sqrt{1 - \mathrm{e}^{nu}}} du = \frac{2}{n} \int_{\mathrm{e}^{ns/2}}^1 \frac{d\varrho_c}{\sqrt{1 - \varrho_c^2}} = \frac{2}{n}\left(\frac{\pi}{2} - \arcsin(\mathrm{e}^{ns/2})\right),$$

which gives that $\phi_c(s) > -\pi/n \, \forall s \in (-\infty, 0]$, and that $\phi_c(s) \to -\pi/n$ as $s \to -\infty$.

Therefore, the (f_s, ϕ_c)-graph $\Sigma_2'(c)$ is a W_c-hypersurface of $\mathbb{H}^n \times \mathbb{R}$ with boundary $\mathscr{H}_0 \times \{0\}$. Also, the height function of $\Sigma_2'(c)$ is unbounded below if $c > -n(n-1)$ and, if $c = -n(n-1), \Sigma_2'(c)$ is contained in the slab $\mathbb{H}^n \times (-\pi/2, 0]$, being asymptotic to the horizontal hyperplane $P_{-\pi/2} := \mathbb{H}^n \times \{-\pi/2\}$. Note that $\Sigma_2'(c)$ is nowhere convex, for its principal curvatures are all negative, except for $k_n = \varrho' > 0$. In addition, the tangent spaces of $\Sigma_2'(c)$ along its boundary are all vertical, for $\phi_c'(s) \to +\infty$ as $s \to 0$, and $\Sigma_2'(c)$ extends C^2-smoothly to $\partial\Sigma_2'(c)$, for $\varrho_c'(0) > 0$.

We conclude from the above considerations that the hypersurface $\Sigma_2(c)$ obtained by the union of the closure of $\Sigma_2'(c)$ with its reflection with respect to the hyperplane $P_0 := \mathbb{H}^n \times \{0\}$ is a properly embedded hypersurface of $\mathbb{H}^n \times \mathbb{R}$ which is invariant by parabolic translations, since the vertical projections of its horizontal sections over $\mathbb{H}^n \times \{0\}$ are horospheres all centered at the same point at infinity (Figure 4).

Finally, we observe that, since $b_c/n \to -1$ as $c \to 0$, given $\epsilon_0 > 0$, there exists $c_0 > 0$ such that

$$\left|\frac{b_c}{n} + 1\right| < \frac{\epsilon_0}{2} \ \ \forall c \in (0, c_0).$$

Hence, from Equation (6.19), one has

$$|\tau_c(s) - 1| \leq \left|\frac{b_c}{n} + 1\right|\mathrm{e}^{ns} + \left|\frac{b_c}{n} + 1\right| < \epsilon_0 \ \ \forall s \in (-\infty, 0), \ \forall c \in (0, c_0),$$

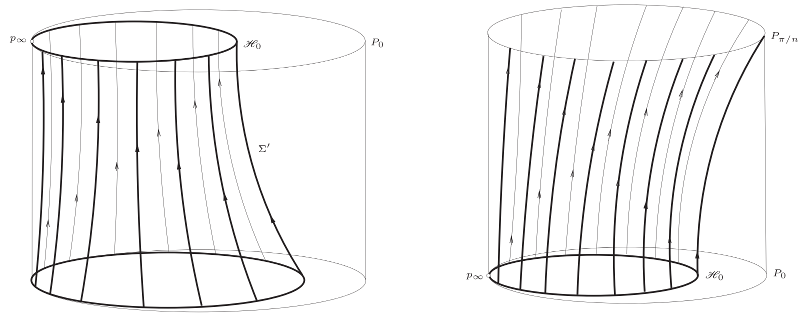

FIGURE 4 Half of the constant scalar curvature (CSC) Weingarten hypersurfaces of Theorem 6.5(ii), where the one on the left is supposed to be unbounded below. (Note we are considering the Poincaré ball model of $\mathbb{H}^n$.)

which implies that, as $c \to 0, \tau_c$ converges uniformly to the constant function $\tau = 1$ on $(-\infty, 0]$. Since $\partial\Sigma_2'(c) = \mathscr{H}_0\, \forall c \in [-n(n-1), 0)$, likewise the hypersurfaces $\Sigma_1(c)$ above, $\Sigma_2(c)$ converges uniformly (on compact sets) to the cylinder $\mathscr{H}_0 \times \mathbb{R}$ over the horosphere $\mathscr{H}_0 = f_0(\mathbb{R}^{n-1}) \subset \mathbb{H}^n$ as $c \to 0$. Note that $T = \partial_t$ is a principal direction of $\mathscr{H}_0 \times \mathbb{R}$ whose corresponding principal curvature vanishes identically. Besides, its other principal curvatures are all equal to 1, and the corresponding principal directions are all tangent to $\mathscr{H}_0$. Thus, by Gauss equation (6.1), $\mathscr{H}_0 \times \mathbb{R}$ is flat, that is, has vanishing sectional curvature everywhere.

Therefore, we have the following result.

Theorem 6.5. *Given $n \geq 3$ and $c \in [-n(n-1), 0)$, there are two properly embedded W_c-hypersurfaces $\Sigma_1(c)$ and $\Sigma_2(c)$ in $\mathbb{H}^n \times \mathbb{R}$ which are homeomorphic to $\mathbb{R}^n$ and invariant by parabolic translations. In addition, they have the following properties:*

- *i) $\Sigma_1(c)$ is a convex (nowhere strictly convex) entire graph over $\mathbb{H}^n$ with constant sectional curvature $K = c/(n(n-1)) \in [-1, 0)$ and constant angle function. For $c = -n(n-1), \Sigma_1(c)$ is a totally geodesic horizontal hyperplane of $\mathbb{H}^n \times \mathbb{R}$ of constant sectional curvature $K = -1$.*
- *ii) $\Sigma_2(c)$ is nowhere convex and symmetric with respect to $\mathbb{H}^n \times \{0\}$. If $c > -n(n-1)$, the height function of $\Sigma_2(c)$ is unbounded, and if $c = -n(n-1)$, $\Sigma_2(c)$ is contained in the slab $\mathbb{H}^n \times (-\pi/n, \pi/n)$, being asymptotic to the horizontal hyperplanes $P_{-\pi/n}$ and $P_{\pi/n}$.*

Furthermore, as $c \to 0$, both $\Sigma_1(c)$ and $\Sigma_2(c)$ converge uniformly (on compact sets) to a flat cylinder $\mathscr{H}_0 \times \mathbb{R}$ over a horosphere $\mathscr{H}_0$ of $\mathbb{H}^n$.

Now, we proceed to construct properly embedded W_c-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ which are invariant by hyperbolic translations. On that account, consider an isometric immersion $f_0 : \mathbb{R}^{n-1} \to \mathbb{H}^n$ such that $f_0(\mathbb{R}^{n-1})$ is a totally geodesic hyperplane of $\mathbb{H}^n$ and let

$$\mathscr{F} := \{f_s : \mathbb{R}^{n-1} \to \mathbb{H}^n\,;\, s \in (-\infty, +\infty)\}$$

be the parallel family of equidistant hypersurfaces to f_0 in $\mathbb{H}^n$. Each f_s is totally umbilical with principal curvatures $\alpha(s) = -\tanh(s)$. Hence, in this setting, and for $s > 0$, Equation (6.5) takes the form

$$\tau'(s) = a(s)\tau(s) + b(s), \quad s \in (0, +\infty), \tag{6.20}$$

where the functions a and b are given by

$$a(s) = -2\coth(s) - (n-2)\tanh(s) \quad \text{and} \quad b(s) = n\mathfrak{C}_n \coth(s), \tag{6.21}$$

being $\mathfrak{C}_n$ the constant defined in Equation (6.12) for $\epsilon = -1$, that is,

$$\mathfrak{C}_n = \frac{c + n(n-1)}{n(n-1)}\,.$$

As for the construction of the hypersurfaces in Theorem 6.4, we look at solutions τ of (6.20) satisfying $\tau(\lambda) = 1$ and $\tau'(\lambda) = a(\lambda) + b(\lambda) < 0$ for suitable values of $s_0 = \lambda$. This last inequality is equivalent to

$$(n\mathfrak{C}_n - 2)\coth^2(\lambda) < n - 2, \tag{6.22}$$

which is valid for all $\lambda > 0$ if $n\mathfrak{C}_n \le 2$. Otherwise, assuming $-n(n-1) \le c < 0$, we have that $(n-2)/(n\mathfrak{C}_n - 2) > 1$. Therefore, setting

$$\lambda_0 = \lambda_0(c) = \operatorname{arctanh}\sqrt{\mathfrak{C}_n},$$

and defining the interval I_c as

$$I_c := \begin{cases} (0, +\infty) & \text{if} \quad n\mathfrak{C}_n \le 2 \\ [\lambda_0, +\infty) & \text{if} \quad n\mathfrak{C}_n > 2, \end{cases}$$

we have that the inequality (6.22) holds for any $\lambda \in I_c$, so that $\tau'(\lambda) < 0$ if $\tau(\lambda) = 1$.

With this notation, we have the following result.

Theorem 6.6. *For any $c \in [-n(n-1), 0)$, there exists a one-parameter family*

$$\mathcal{S} = \{\Sigma(\lambda)\,;\, \lambda \in I_c\}$$

of properly embedded nowhere convex W_c-hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ which are homeomorphic to $\mathbb{R}^n$ and invariant by hyperbolic translations. Each $\Sigma(\lambda) \in \mathcal{S}$ is symmetric with respect to $\mathbb{H}^n \times \{0\}$ and has the following additional properties:

i) If $c > -n(n-1)$, the height function of Σ is unbounded above and below, and $\Sigma(\lambda_0)$ has constant sectional curvature $K = \frac{c}{n(n-1)} \in (-1, 0)$.

ii) If $c = -n(n-1)$, Σ is contained in a slab $\mathbb{H}^n \times (-\sigma\pi/n, \sigma\pi/n)$, $\sigma \in (0, 1]$, and is asymptotic to both horizontal hyperplanes $P_{-\sigma\pi/n}$ and $P_{\sigma\pi/n}$.

Proof. Given $\lambda \in I_c$, we have from Equation (6.10) that the solution τ_λ of Equation (6.20) satisfying $\tau_\lambda(\lambda) = 1$ is given by

$$\tau_\lambda(s) = \mathfrak{C}_n \frac{\cosh^n(s) - \cosh^n(\lambda)}{\cosh^{n-2}(s)\sinh^2(s)} + \frac{\cosh^{n-2}(\lambda)\sinh^2(\lambda)}{\cosh^{n-2}(s)\sinh^2(s)}, \quad s \in [\lambda, +\infty). \tag{6.23}$$

By the definition of I_c, τ_λ is decreasing near λ. Also, from Equation (6.23), τ_λ is positive in $[\lambda, +\infty)$. Let us see that τ_λ has no critical points in this interval. Assuming otherwise, let $s_1 > \lambda$ be such that $\tau_\lambda'(s_1) = 0$. Then, from Equation (6.20), we have $\tau_\lambda(s_1) = -b(s_1)/a(s_1)$, which gives

$$2\tau_\lambda(s_1) - n\mathfrak{C}_n = n\mathfrak{C}_n\left(\frac{2\coth(s_1)}{2\coth(s_1) + (n-2)\coth(s_1)} - 1\right) < 0.$$

Therefore,

$$\tau_\lambda''(s_1) = a'(s_1)\tau_\lambda(s_1) + b'(s_1) = \frac{2\tau_\lambda(s_1) - n\mathfrak{C}_n}{\sinh^2(s)} - \frac{n-2}{\cosh^2(s)} < 0,$$

which implies that any critical point of τ_λ in $[\lambda, +\infty)$ is a maximum. However, since τ_λ is decreasing near λ, a local maximum point of τ_λ should be preceded by a minimum point. Thus, τ_λ has no critical points, so that it is decreasing in $[\lambda, +\infty)$.

Therefore, the function $\varrho_\lambda = \sqrt{\tau_\lambda}$ is well defined in $[\lambda, +\infty)$ and satisfies:

$$0 < \varrho_\lambda(s) \le 1 = \varrho_\lambda(\lambda) \quad \text{and} \quad \varrho_\lambda'(s) < 0 \ \ \forall s \in [\lambda, +\infty).$$

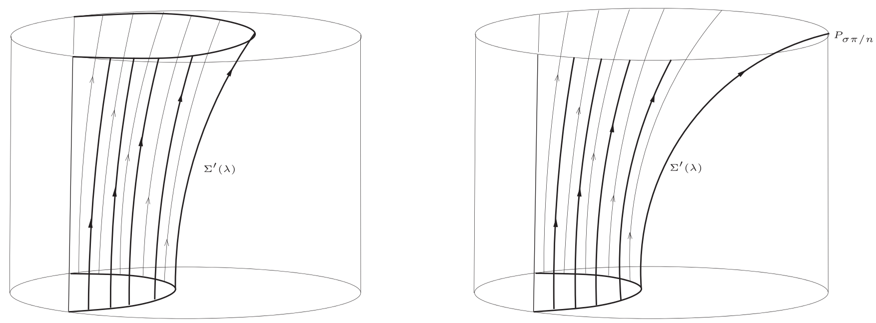

FIGURE 5 A piece of the W_c-hypersurface of Theorem 6.6 (case (i) on the left, case (ii) on the right).

Hence, the associated ϕ-function

$$\phi_\lambda(s) = \int_\lambda^s \frac{\varrho_\lambda(u)}{\sqrt{1-\varrho_\lambda^2(u)}}du, \quad s \in [\lambda, +\infty),$$

is well defined and satisfies $\phi'_\lambda(s) \to +\infty$ as $s \to \lambda$. This, as before, gives that the corresponding (f_s, ϕ)-graph $\Sigma'(\lambda)$ is a W_c-hypersurface of $\mathbb{H}^n \times \mathbb{R}$ with boundary $f_\lambda(\mathbb{R}^{n-1}) \times \{0\}$. So, proceeding as in the previous proofs, we obtain a properly embedded W_c-hypersurface $\Sigma(\lambda)$ in $\mathbb{H}^n \times \mathbb{R}$ by reflecting $\Sigma'(\lambda)$ with respect to the horizontal hyperplane $\mathbb{H}^n \times \{0\}$.

To prove the assertions (i) and (ii), let us first observe that Equation (6.23) yields

$$\lim_{s \to +\infty} \varrho_\lambda(s) = \sqrt{\mathfrak{C}_n}\ . \tag{6.24}$$

Hence, if $\mathfrak{C}_n > 0$,

$$\phi_\lambda(s) = \int_\lambda^s \frac{\varrho_\lambda(u)}{\sqrt{1-\varrho_\lambda^2(u)}}du \geq \int_\lambda^s \varrho_\lambda(u)du \geq \sqrt{\mathfrak{C}_n}(s-\lambda),$$

which implies that ϕ_λ is unbounded. Besides, setting $c = Kn(n-1)$, we have $\lambda_0 = \operatorname{arctanh}\sqrt{1+K}$. In this case, it is easily checked that

$$\varrho_{\lambda_0}(s) = (1+K)\coth(s), \quad s \in [\lambda_0, +\infty).$$

However, ϱ_{λ_0} is also a solution of Equation (6.7) (for $\alpha(s) = \tanh(s)$), which implies that $\Sigma(\lambda_0)$ has constant sectional curvature K. This proves (i).

If $\mathfrak{C}_n = 0$, we have that $\tau'_\lambda = a\tau_\lambda$, which implies that $2\varrho'_\lambda = a\varrho_\lambda$. Thus, observing that $\sup(-1/a) = 1/n$, we have

$$\phi_\lambda(s) = \int_\lambda^s \frac{2\varrho'_\lambda(u)}{a(u)\sqrt{1-\varrho_\lambda^2(u)}}du \leq \frac{2}{n}\int_{\varrho_\lambda(s)}^1 \frac{d\varrho_\lambda}{\sqrt{1-\varrho_\lambda^2}} = \frac{2}{n}\left(\frac{\pi}{2} - \arcsin(\varrho_\lambda(s))\right),$$

which implies that $\phi_\lambda(s) < \pi/n$. In particular, there exists $\sigma \in (0, 1]$ such that

$$\sup \phi_\lambda = \sigma\frac{\pi}{n},$$

which shows (ii) and concludes the proof (Figure 5). □

7 | UNIQUENESS OF ROTATIONAL ELLIPTIC WEINGARTEN SPHERES

As we have pointed out in Remark 4.2, the maximum principle applies to elliptic Weingarten hypersurfaces of $M \times \mathbb{R}$. This fact, together with the main results in [7] (see also [15, 23]), allows us to apply the Alexandrov reflection method to provide uniqueness results for the rotational elliptic Weingarten spheres of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ ($\epsilon \neq 0, n \geq 3$) we constructed in Section 5. Similar results were obtained in [8] for hypersurfaces of constant higher order mean curvatures.

Definition 7.1. Let P_0 be a totally geodesic hypersurface of either hyperbolic space $\mathbb{H}^n$ or an open hemisphere $\mathbb{S}^n_+$ of $\mathbb{S}^n$. We call the hypersurface $P := P_0 \times \mathbb{R}$ a *vertical hyperplane* of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$.

Regarding the spherical part of the above definition, we remark that the Alexandrov method is not effective for vertical hyperplanes over the whole sphere. However, it works well for vertical hyperplanes over open hemispheres (cf. [1], p. 144).

Theorem 7.2 (Jellett–Liebmann-type theorem). *For $n \geq 3$, let Σ be a compact connected elliptic Weingarten hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$. Then, Σ is an embedded rotational sphere.*

Proof. Since Σ is strictly convex and compact, [7, Theorems 1 and 2] apply and give that Σ is embedded and homeomorphic to $\mathbb{S}^n$. In this way, we can perform Alexandrov reflections on Σ with respect to horizontal hyperplanes $P_t := \mathbb{Q}^n_\epsilon \times \{t\}$ coming down from above Σ. Then, a standard argument using the maximum principle shows that Σ is symmetric with respect to some horizontal plane P_{t_0}.

For $\epsilon = -1$, we can proceed as above by performing Alexandrov reflections on Σ with respect to vertical hyperplanes of $\mathbb{H}^n \times \mathbb{R}$ to conclude that it has vertical symmetries in all directions. Therefore, Σ is rotational.

For $\epsilon = 1$, assuming $t_0 = 0$ and identifying $\mathbb{S}^n \times \{0\}$ with $\mathbb{S}^n$, we have that $\Sigma_0 := \Sigma \cap \mathbb{S}^n$ is the boundary of $\pi(\Sigma)$ in $\mathbb{S}^n$ and that $\Sigma \setminus \Sigma_0$ has two connected components, each of which being a graph over $\mathrm{int}(\pi(\Sigma))$. By [7, Lemma 1], the second fundamental form of Σ_0, as a hypersurface of $\mathbb{S}^n$, is positive definite. In particular, Σ_0 is not totally geodesic in $\mathbb{S}^n$. Thus, by [11, Theorem 1], Σ_0 is contained in an open hemisphere $\mathbb{S}^n_+$ of $\mathbb{S}^n$, which implies that the same is true for $\pi(\Sigma)$. Indeed, the other option would be $\mathbb{S}^n \setminus \pi(\Sigma) \subset \mathbb{S}^n_+$, in which case Σ would have at least one concave point, a contradiction with the assumption it is strictly convex. Hence, $\Sigma \subset \mathbb{S}^n_+ \times \mathbb{R}$, and again we may apply Alexandrov reflections on the vertical hyperplanes in $\mathbb{S}^n_+ \times \mathbb{R}$ to deduce that Σ is rotational. □

Let us see now that the compactness hypothesis in Theorem 7.2 can be replaced by completeness if we add conditions on the height function ξ of Σ and on its second fundamental form. This is accomplished by means of the following general height estimate, obtained in [8].

Lemma 7.3 [8, Proposition 3]. *Consider an arbitrary Riemannian manifold M, and let $\Sigma \subset M \times \mathbb{R}$ be a compact vertical graph of a nonnegative function defined on a domain $\Omega \subset M \times \{0\}$. Assume Σ strictly convex up to $\partial\Sigma \subset M \times \{0\}$. Under these conditions, the following height estimate holds:*

$$\xi(x) \leq \frac{1}{\inf_\Sigma k_0} \quad \forall x \in \Sigma, \tag{7.1}$$

where k_0 is the least principal curvature function of Σ.

Theorem 7.4. *Assume $n \geq 3$, and let Σ be a complete connected elliptic Weingarten hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ whose height function has a local extreme point $x \in \Sigma$. If the least principal curvature k_0 of Σ is bounded away from zero, then Σ is an embedded rotational sphere.*

Proof. As in Theorem 7.2, Σ fulfills the hypotheses of [7, Theorems 1 and 2], which implies that Σ is properly embedded and homeomorphic to either $\mathbb{S}^n$ or $\mathbb{R}^n$. In the former case, the result follows from Theorem 7.2, so we can assume that Σ is noncompact. Under this assumption, [7, Theorems 1 and 2] also give that the extreme point x (which we assume to be a maximum) is unique, and that the height function ξ of Σ is unbounded (below).

Given a horizontal hyperplane $P_t = \mathbb{Q}^n_\epsilon \times \{t\}$ with $t < \xi(x)$, the part Σ^+_t of Σ which lies above P_t must be a vertical graph with boundary in P_t. If not, for some t' between t and $\xi(x)$, $P_{t'}$ would be orthogonal to Σ at one of its points. Then, the boundary maximum principle would give that Σ is symmetric with respect to $P_{t'}$, which is impossible, since we are assuming ξ unbounded, and the closure of $\Sigma^+_{t'}$ in Σ is compact.

It follows from the above that, for $|t|$ sufficiently large, one has

$$\xi(x) - t > \frac{1}{\inf_{\Sigma} k_0} \geq \frac{1}{\inf_{\Sigma_t^+} k_0},$$

which clearly contradicts Lemma 7.3. This finishes the proof. □

Corollary 7.5. *Any complete strictly convex CSC hypersurface of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ $(n \geq 3)$ satisfying the hypotheses of Theorem 7.4 is necessarily an embedded rotational sphere of constant sectional curvature $K > \epsilon$.*

Proof. Let $\Sigma_0 \subset \mathbb{Q}_\epsilon^n \times \mathbb{R}$ be a strictly convex hypersurface of CSC c (and so an elliptic W_c-hypersurface) which fulfills the hypotheses of Theorem 7.4. Then, Σ_0 is an embedded rotational W_c-sphere. Let $o \in \Sigma_0$ be the point of least height on Σ_0. Clearly, $\Theta^2 = 1$ at o. This, together with Equation (6.2) and the strict convexity of Σ_0, implies that $c > \epsilon n(n-1)$. Also, since Σ_0 is rotational, in a neighborhood of o, Σ_0 is a rotational W_c-(f_s, ϕ)-graph. In particular, its ϱ function is the solution of Equation (6.4), for $\alpha = -\cot_\epsilon$, which satisfies $\varrho(0) = 0$. Thus, in a neighborhood of o, up to an ambient isometry, Σ_0 coincides with the hypersurface Σ of CSC $c > \epsilon n(n-1)$ obtained in Theorem 6.3. Since Σ_0 and Σ are both elliptic, they must coincide. In particular, $\Sigma = \Sigma_0$ is compact, that is, $c > 0$ and Σ_0 is a rotational sphere of constant curvature $K = c/(n(n-1)) > \epsilon$. □

8 | CONSTANT SECTIONAL CURVATURE HYPERSURFACES OF $\mathbb{Q}_\epsilon^n \times \mathbb{R}$

In this final section, we apply the methods and results developed in the previous ones to give new proofs of the main theorems of [22]. There, for $n \geq 3$, the authors construct and classify the constant sectional curvature hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ $(\epsilon \neq 0)$. Their proofs rely on parameterizations of symmetric hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$, as introduced in [10], as well as on the main result of that paper. Our proofs, instead, are coordinate-free and more direct. Another distinction of our approach is the simple way we show that constant sectional curvature hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ with nonvanishing T-field have the T-property. This fact, which is also proved and nicely used in [22], plays a fundamental role here, as we shall see.

First, let us observe that, if $\Sigma := \Sigma_0 \times \mathbb{R}$ is a symmetric cylinder over a hypersurface Σ_0 of $\mathbb{Q}_\epsilon^n$, then Σ_0 is an open set of a geodesic sphere, a horosphere or an equidistant hypersurface. Thus, from Gauss equation (6.1), we have that Σ has constant sectional curvature if and only if Σ_0 is contained in a horosphere. If so, Σ is necessarily flat. We also point out that, by the first equality in Equation (6.6), there is no flat parabolic (f_s, ϕ)-graph in $\mathbb{Q}_\epsilon^n \times \mathbb{R}$.

Theorem 8.1. *Given $n \geq 3$, let Σ be a connected symmetric hypersurface of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ with constant sectional curvature K. Then, $K \geq \epsilon$ and Σ is an open set of one of the following properly embedded hypersurfaces:*

- *(i) One of the rotational hypersurfaces of constant sectional curvature $K > \epsilon$ of Theorem 6.3.*
- *(ii) One of the translational hypersurfaces of constant sectional curvature $K \in (-1, 0)$ of Theorems 6.5-(i) and 6.6-(i).*
- *(iii) A horizontal hyperplane of constant sectional curvature $K = \epsilon$.*
- *(iv) A flat vertical cylinder over a horosphere.*

Proof. Let us suppose that ΘT never vanishes on Σ. In this case, Σ is given by a union of vertical graphs with no critical points. Let Σ' be one of these graphs. Then, since Σ is symmetric, up to a reflection over a horizontal hyperplane, Σ' is an (f_s, ϕ)-graph over a family $\mathscr{F}$ of parallel totally umbilical hypersurfaces of $\mathbb{Q}_\epsilon^n$ (recall that ϕ is supposed to be increasing).

As we have seen in Section 6, the ϱ function of Σ' satisfies the ODE

$$\alpha\varrho\varrho' + (\alpha^2 + \epsilon)\varrho^2 = 0, \tag{8.1}$$

where $\alpha(s)$ is the principal curvature of f_s. Also, from the first equality in Equation (6.6), we have that $K > \epsilon$ and that any initial condition $\varrho_0 = \varrho(s_0)$ is determined by K and $\alpha(s_0)$.

Since Σ is connected, it is either rotational or translational. Assuming the former, we have that Σ' is rotational, so that $\alpha = -\coth_\epsilon$ and the solution ϱ of Equation (8.1) is $\varrho(s) = C\tan_\epsilon(s)$ for some constant $C > 0$. However, since C is determined by K and $\alpha(s_0)$, this solution must coincide with the one defined in Equation (6.14), that is,

$$\varrho(s) = (K - \epsilon)\tan_\epsilon(s).$$

In particular, up to an ambient isometry, Σ' is contained in the properly embedded CSC hypersurface of Theorem 6.3, say $\tilde{\Sigma}$, that has constant sectional curvature K. (Note that $\tilde{\Sigma}$ is built on an (f_s, ϕ)-graph whose ϱ-function is $\tilde{\varrho}(s) = (K - \epsilon)\tan_\epsilon(s) = \varrho(s)$.) Since Σ' is arbitrary and Σ is connected, we have that $\Sigma \subset \tilde{\Sigma}$.

If Σ is translational, it is either parabolic or hyperbolic. In any case, we can argue as in the preceding paragraph and conclude that Σ is contained in one of the translational hypersurfaces of Theorems 6.5-(i) and 6.6-(i).

Now, suppose that T vanishes on an open set $\mathcal{O}$ of Σ. Thus, $\mathcal{O}$ is contained in a (totally geodesic) horizontal hyperplane. The Gauss equation then gives that $\mathcal{O}$, and so Σ, has constant sectional curvature $K = \epsilon$. This implies that $\Sigma = \mathcal{O}$. Otherwise, there would be a point in Σ at which $\Theta T \neq 0$. But, as we have seen, $K > \epsilon$ at such a point.

Analogously, assume that Θ vanishes on an open set $\mathcal{O}$ of Σ. In this case, as we pointed out, $\mathcal{O}$ is a flat cylinder over an open set of a horosphere of $\mathbb{H}^n$. In particular, Σ is flat and parabolic. Since a parabolic (f_s, ϕ)-graph cannot be flat, ΘT must vanish on Σ, which implies that $\Sigma = \mathcal{O}$.

Finally, let us suppose that neither T nor Θ vanishes on an open set of Σ. Under this assumption, the set $\mathcal{O} \subset \Sigma$ on which ΘT never vanishes is open and dense in Σ. However, from the first part of the proof, any connected component of $\mathcal{O}$ is contained in a fixed hypersurface $\tilde{\Sigma}$ (from one of the Theorems 6.3, 6.5-(i) or 6.6-(i)), which implies that the same is true for Σ. This concludes the proof. □

Given a hypersurface Σ of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$, let $\{X_1, \dots, X_n\} \subset T\Sigma$ be an orthonormal frame of its principal directions. It follows from Gauss equation (2.3) that

$$\mathrm{Ric}(X_i, X_j) = \sum_{k=1}^{n} \langle \bar{R}(X_k, X_i)X_j, X_k\rangle + H\delta_{ij}k_i - \delta_{ij}k_i^2, \tag{8.2}$$

where $k_1, \dots, k_n$ are the corresponding principal curvatures of Σ, and Ric denotes its *Ricci tensor*, which we define as

$$\mathrm{Ric}(X, Y) := \mathrm{trace}(Z \mapsto R(Z, X)Y), \quad X, Y \in T\Sigma.$$

The following lemma, which has its own interest, gives a simple characterization of the hypersurfaces of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ having the T-property. (Recall that a hypersurface $\Sigma \subset M \times \mathbb{R}$ is said to have the T-property if T is a principal direction at any of its points.)

Lemma 8.2. *Given $n \geq 3$, let Σ be a hypersurface of $\mathbb{Q}^n_\epsilon \times \mathbb{R}$ with nonvanishing T-field. Then, Σ has the T-property if and only if its principal directions $X_1, \dots, X_n$ diagonalize its Ricci tensor. Consequently, if Σ is an Einstein hypersurface (in particular, if Σ has constant sectional curvature), then it has the T-property.*

Proof. Choosing $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k \neq i$, it follows from Equation (2.4) that

$$\langle \bar{R}(X_k, X_i)X_j, X_k\rangle = -\epsilon\langle X_i, T\rangle\langle X_j, T\rangle.$$

Combining this equality with Equation (8.2), we get

$$\mathrm{Ric}(X_i, X_j) = \sum_{k=1}^{n} \langle \bar{R}(X_k, X_i)X_j, X_k\rangle = -\epsilon(n-2)\langle X_i, T\rangle\langle X_j, T\rangle,$$

from which the result follows. □

Let us show now that, setting

$$I_\epsilon := \begin{cases} (0, 1) & \text{if } \epsilon = 1 \\ (-1, 0) & \text{if } \epsilon = -1, \end{cases}$$

for all $K \in I_\epsilon$, there exists a nonsymmetric constant sectional curvature hypersurface Σ_K in $\mathbb{Q}_\epsilon^3 \times \mathbb{R}$ whose angle function is constant, and whose sectional curvature is K (see also [22, Section 6]). For its construction, one has just to consider a parallel family $\mathscr{F}$ of embeddings $f_s : M_0 \to \mathbb{Q}_\epsilon^3$, $s \in (-\delta, \delta)$, such that f_0 is nonumbilical and flat with the induced metric, that is, $k_1^0(p)k_2^0(p) = -\epsilon$ for all $p \in M_0$. Then, the classical formula for the principal curvatures of the parallel hypersurfaces f_s gives that all f_s are, in fact, flat and nonumbilical, so that

$$k_1^s(p)k_2^s(p) = -\epsilon \;\; \forall p \in M_0, \;\; s \in (-\delta, \delta).$$

Now, consider the (f_s, ϕ)-graph Σ_K with constant ϱ-function $\varrho = \sqrt{1 - \epsilon K}$. Then, Σ_K has constant angle $\Theta = \sqrt{1 - \varrho^2}$. Besides, the principal curvatures of Σ_K are $k_1 = -\varrho k_1^s$, $k_2 = -\varrho k_2^s$, and $k_3 = 0$. Hence, from Gauss equation (6.1), Σ_K has indeed constant sectional curvature K.

Definition 8.3. We shall call such a $\Sigma_K \subset \mathbb{Q}_\epsilon^3 \times \mathbb{R}$ a *flat-foliated graph.*

Our next and final result, together with Theorem 8.1, provides a full classification of the constant sectional curvature hypersurfaces of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ $(n \geq 3)$.

Theorem 8.4. *Let Σ be a connected hypersurface of $\mathbb{Q}_\epsilon^n \times \mathbb{R}$ with constant sectional curvature K. If $n > 3$, Σ is symmetric. If $n = 3$, Σ is either symmetric or a nonsymmetric hypersurface which is locally a constant angle flat-foliated graph Σ_K.*

Proof. Proceeding as in the proof of Theorem 8.1, let us assume first that ΘT is nowhere vanishing on Σ, in which case Σ is a union of vertical graphs with no critical points.

Let $\Sigma' \subset \Sigma$ be a vertical graph. By Lemma 8.2, T is a principal direction of Σ'. Hence, [26, Theorem 1] (see also [9, Theorem 6]) applies and gives that Σ' is an (f_s, ϕ)-graph over a family $\mathscr{F}$ of parallel hypersurfaces of $\mathbb{Q}_\epsilon^n$. Thus, by combining the identities (3.7) with Gauss equation (6.1), we conclude that the ϱ function of Σ' satisfies

$$\begin{aligned} K &= k_i^s(p)k_j^s(p)\varrho^2(s) + \epsilon \quad (i \neq j = 1, \dots, n-1). \\ K &= -k_i^s(p)\varrho(s)\varrho'(s) + \epsilon(1 - \varrho^2(s)) \quad (i = 1, \dots, n-1). \end{aligned} \tag{8.3}$$

If $n > 3$, it follows easily from the first of the equations in Equation (8.3) that $k_i^s(p)$ is independent of i and p. Hence, each $f_s \in \mathscr{F}$ is totally umbilical, which implies that Σ' is symmetric. Therefore, Σ is symmetric, since $\Sigma' \subset \Sigma$ is arbitrary.

Suppose now that $n = 3$. If $f_s \in \mathscr{F}$ is totally umbilical for any graph $\Sigma' \subset \Sigma$, as above, we have that Σ is symmetric. So, assume that there is $\Sigma' \subset \Sigma$ such that f_s is nontotally umbilical, so that Σ', and so Σ, is nonsymmetric. In this case, the ϱ function of Σ' is constant. Otherwise, from the second equation in Equation (8.3), f_s would be totally umbilical. From this same equation, we have that Σ' has constant sectional curvature $K = \epsilon(1 - \varrho^2) = \epsilon\Theta^2$. So, from the first equality in Equation (8.3), we have $k_i^s(p)k_j^s(p) = -\epsilon$, which implies that each f_s is flat, that is, Σ' is a constant angle flat-foliated graph Σ_K.

The above reasoning shows that, in a neighborhood of a point at which $\Theta T \neq 0$, Σ is either a symmetric (f_s, ϕ)-graph or a flat-foliated graph. In the former case, as we know, $K > \epsilon$. In the latter case, $K = \epsilon\Theta^2 \neq \epsilon$. Therefore, we can argue just as in the proof of Theorem 8.1 to show that Σ is contained in a horizontal hyperplane if T vanishes on an open set of Σ. Since neither parabolic (f_s, ϕ)-graphs nor flat-foliated graphs are flat, we can also argue as in the proof of Theorem 8.1 to show that Σ is contained in a cylinder over a horosphere if Θ vanishes on an open set of Σ.

Finally, if neither T nor Θ vanishes in an open set Σ, the set $\mathcal{O} \subset \Sigma$ of points, where ΘT never vanishes is open and dense in Σ. Since the theorem is valid for any connected component of $\mathcal{O}$, it follows that it is valid for Σ as well. □

ACKNOWLEDGMENTS

The second author was partially supported by CNPq/Brazil. We are in debt to A. Martinez for letting us know about Gálvez and Mira paper [18], as well as for suggesting to us bringing (a part of) it to the n-dimensional context. We are also grateful to J. A. Gálvez, who read a preliminary version of the paper and made valuable suggestions. Finally, we thank the referees for the careful reading and insightful comments.

ORCID

R. F. de Lima https://orcid.org/0000-0002-7331-2721
J. P. dos Santos https://orcid.org/0000-0002-4102-8726

REFERENCES

[1] U. Abresch and H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in* $\mathbb{S}^2\times\mathbb{R}$ *and* $\mathbb{H}^2\times\mathbb{R}$, Acta Math. **193** (2004), 141–174.
[2] J. A. Aledo, J. M. Espinar, and J. A. Gálvez, *Complete surfaces of constant curvature in* $\mathbb{H}^2\times\mathbb{R}$ *and* $\mathbb{S}^2\times\mathbb{R}$, Calc. Var. Partial Differ. Equations. **29** (2007), 347–363.
[3] J. A. Aledo, J. M. Espinar, and J. A. Gálvez, *Surfaces with constant curvature in* $M^2\times\mathbb{R}$. *Height estimates and representation*, Bull. Braz. Math. Soc. **38** (2007), 533–554.
[4] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large I*, Amer. Math. Soc. Transl. Ser. 2 **21** (1962), 341–354.
[5] S. S. Chern, *On special W-surfaces*, Proc. Amer. Math. Soc. **6** (1955), 783–786.
[6] B. Daniel, *Isometric immersions into* $\mathbb{S}^n\times\mathbb{R}$ *and* $\mathbb{H}^n\times\mathbb{R}$ *and applications to minimal surfaces*, Trans. Amer. Math. Soc. **361** (2009), no. 12, 6255–6282.
[7] R. F. de Lima, *Embeddedness, convexity, and rigidity of hypersurfaces in product spaces*, Ann. Glob. Anal. Geom. **59** (2021), 319–344.
[8] R. F. de Lima, F. Manfio, and J. P. dos Santos, *Hypersurfaces of constant higher order mean curvature in* $M\times\mathbb{R}$, Ann. Mat. **201** (2022), 2979–3028. https://doi.org/10.1007/s10231-022-01229-3
[9] R. F. de Lima and P. Roitman, *Helicoids and catenoids in* $M\times\mathbb{R}$, Ann. Mat. Pura Appl. **200** (2021), 2385–2421.
[10] F. Dillen, J. Fastenakels, and J. Van der Veken, *Rotation hypersurfaces in* $\mathbb{S}^n\times\mathbb{R}$ *and* $\mathbb{H}^n\times\mathbb{R}$, Note Mat. **29** (2008), 41–54.
[11] M. do Carmo and F. Warner, *Rigidity and convexity of hypersurfaces in spheres*, J. Differ. Geom. **4** (1970), 133–144.
[12] M. Domínguez-Vázquez, *An introduction to isoparametric foliations*, preprint (2018). http://xtsunxet.usc.es/miguel/teaching/jae2018.html
[13] J. M. Espinar, J. A. Gálvez, and H. Rosenberg, *Complete surfaces with positive extrinsic curvature in product spaces*, Comment. Math. Helv. **84** (2009), 351–386.
[14] J. M. Espinar and I. de Oliveira, *Locally convex surfaces immersed in a Killing submersion*, Bull. Braz. Math. Soc., New Series **44** (2013), 1–17.
[15] J. M. Espinar and H. Rosenberg, *When strictly locally convex hypersurfaces are embedded*, Math. Z. **271** (2012), 1075–1090.
[16] A. Folha and C. Peñafiel, *Weingarten type surfaces in* $\mathbb{H}^2\times\mathbb{R}$ *and* $\mathbb{S}^2\times\mathbb{R}$, Canad. J. Math. **69** (2017), 1292–1311.
[17] J. A. Gálvez and P. Mira, *Uniqueness of immersed spheres in three-manifolds*, J. Differ. Geom. **116** (2020), no. 3, 459–480.
[18] J. A. Gálvez and P. Mira, *Rotational symmetry of Weingarten spheres in homogeneous three-manifolds*, J. Reine Angew. Math. **773** (2021), 21–66.
[19] C. Gerhardt, *Closed Weingarten hypersurfaces in Riemannian manifolds*, J. Differ. Geom. **43** (1996), 612–641.
[20] P. Hartman and A. Wintner, *Umbilical points and W-surfaces*, Amer. J. Math. **76** (1954), 502–508.
[21] J. Kazdan, *Unique continuation in geometry*, Commun. Pure Appl. Math. **XLI** (1988), 667–681.
[22] F. Manfio and R. Tojeiro, *Hypersurfaces with constant sectional curvature of* $\mathbb{S}^n\times\mathbb{R}$ *and* $\mathbb{H}^n\times\mathbb{R}$, Illinois J. Math. **55** (2011), 397–415.
[23] I. de Oliveira and S. Schweitzer, *Locally convex hypersurfaces immersed in* $H^n\times\mathbb{R}$, Geom Dedicata. **188** (2017), 17–32.
[24] M. Protter, *Unique continuation for elliptic equations*, Trans. Amer. Math. Soc. **95** (1960), 81–91.
[25] H. Rosenberg and R. Sa Earp, *The geometry of properly embedded special surfaces in* $\mathbb{R}^3$*; e.g. surfaces satisfying* $aH + bK = 1$*, where a and b are positive*, Duke Math. J. **73** (1994), 291–306.
[26] R. Tojeiro, *On a class of hypersurfaces in* $\mathbb{S}^n\times\mathbb{R}$ *and* $\mathbb{H}^n\times\mathbb{R}$, Bull. Braz. Math. Soc. **41** (2010), 199–209.

How to cite this article: R. F. de Lima, A. K. Ramos, and J. P. dos Santos, *Elliptic Weingarten hypersurfaces of Riemannian products*, Math. Nachr. **296** (2023), 4712–4738. https://doi.org/10.1002/mana.202200025