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Convexity, Rigidity, and Reduction of Codimension of Isometric Immersions into Space Forms

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Abstract We consider isometric immersions of complete connected Riemannian manifolds into space forms of nonzero constant curvature. We prove that if such an immersion is compact and has semi-definite second fundamental form, then it is an embedding with codimension one, its image bounds a convex set, and it is rigid. This result generalizes previous ones by do Carmo and Lima, as well as by do Carmo and Warner. It also settles affirmatively a conjecture by do Carmo and Warner. We establish a similar result for complete isometric immersions satisfying a stronger condition on the second fundamental form. We extend to the context of isometric immersions in space forms a classical theorem for Euclidean hypersurfaces due to Hadamard. In this same context, we prove an existence theorem for hypersurfaces with prescribed boundary and vanishing Gauss-Kronecker curvature. Finally, we show that isometric immersions into space forms which are regular outside the set of totally geodesic points admit a reduction of codimension to one.

Keywords Isometric immersion · Convexity · Rigidity · Reduction of codimension

Mathematics Subject Classification Primary 53B02; Secondary 53C42

In Memory of Manfredo do Carmo and Elon Lima.

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1 Introduction

Convexity and rigidity are among the most essential concepts in the theory of submanifolds. In his work, Sacksteder established two fundamental results involving these concepts. Combined, they state that for M^n a non-flat complete connected Riemannian manifold with nonnegative sectional curvatures, any isometric immersion $f\colon M^n \to \mathbb{R}^{n+1}$ is, in fact, an embedding and has $f(M)$ as the boundary of a convex set. In particular, M is diffeomorphic to the Euclidean space $\mathbb{R}^n$ or to the unit sphere S^n (Sacksteder 1960). In the latter case, f is rigid, that is, for any other isometric immersion $g\colon M^n \to \mathbb{R}^{n+1}$, there exists a rigid motion $\Phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $g = \Phi \circ f$ (Sacksteder 1962).

The convexity part of this statement is a Hadamard-Stoker type theorem, since it refers to the results of Hadamard (1897) and Stoker (1936), who considered the compact and complete cases, respectively, assuming $n = 2$ and M with positive curvature everywhere. The rigidity part is a generalization of the classical Cohn-Vossen rigidity theorem for ovaloids.

Naturally, the extension of Sacksteder's results to general isometric immersions into space forms became a matter of interest. However, as shown by the standard immersion of $S^n \times S^n$ into $\mathbb{R}^{2n+2}$, Sacksteder Theorem in Sacksteder (1960) is not valid in higher codimension. On the other hand, its proof relies mostly on the semi-definiteness of the second fundamental form of the immersion, which, in codimension one, is equivalent to the assumed non-negativeness of the sectional curvatures of M. Thus, in order to get similar results in higher codimension, it is natural to assume that the second fundamental form is semi-definite, that is, at each point and in any normal direction, all the nonzero eigenvalues of the corresponding shape operator have the same sign.

M. do Carmo and E. Lima established in do Carmo and Lima (1969) a Hadamard-Stoker type theorem for isometric immersions of compact manifolds into Euclidean space with arbitrary codimension. Namely, they proved that if M^n is a compact connected Riemannian manifold and $f\colon M^n \to \mathbb{R}^{n+p}$ is an isometric immersion whose second fundamental form is semi-definite (and definite at one point), then f admits a reduction of codimension to one and embeds M onto the boundary of a compact convex set. Subsequently, this result was extended by Jonker (1975) to complete isometric immersions (see Sect. 5 for a precise statement).

In do Carmo and Warner (1970), M. do Carmo and F. Warner considered compact connected hypersurfaces $f\colon M^n \to S^{n+1}$. They proved that if all sectional curvatures of M are greater than or equal to 1 (the curvature of the ambient space), then f is rigid and embeds M onto the boundary of a compact convex set contained in an open hemisphere of S^{n+1}. In addition, it was shown that, except for the rigidity part, this theorem remains valid if one replaces the sphere S^{n+1} by the hyperbolic space $\mathbb{H}^{n+1}$ and assume that all sectional curvatures of M are no less than -1. The authors also conjectured the rigidity of f for this case.

In the present paper, we extend do Carmo-Lima and do Carmo-Warner theorems to isometric immersions of arbitrary codimension into space forms of nonzero constant curvature. We settle affirmatively, as well, the aforementioned do Carmo and Warner's conjecture. More precisely, we obtain the following result.

Theorem 1 *Let $f: M^n \to Q_c^{n+p}$ be an isometric immersion of a compact connected Riemannian manifold into the space form of constant curvature $c \neq 0$. Assume that f is non-totally geodesic and has semi-definite second fundamental form. Then, f is an embedding of M into a totally geodesic $(n+1)$-dimensional submanifold $Q_c^{n+1} \subset Q_c^{n+p}$, $f(M)$ is the boundary of a compact convex set of Q_c^{n+1}, and f is rigid. In particular, M is diffeomorphic to a sphere.*

As is well known, the flat n-dimensional Clifford torus can be embedded into the hyperbolic space $\mathbb{H}^{2n}$. Additionally, one can easily obtain non-totally geodesic isometric immersions $f: S^n \to S^{2n+1}$ whose codimension cannot be reduced to one (see Dajczer 1990, p. 76). So, Theorem 1 is no longer true if we replace the condition on the second fundamental form by that of M having no sectional curvatures less than c. However, M has this latter property if the immersion $f : M^n \to Q_c^{n+p}$ has semi-definite second fundamental form (see Proposition 3).

Due to the Bonnet-Myers Theorem and the above considerations, we can replace compactness by completeness in the statement of Theorem 1 if $c > 0$. Nevertheless, for $c < 0$, we cannot expect to obtain a Hadamard-Stoker type theorem if we assume that M is complete and noncompact. Indeed, there are complete immersions in hyperbolic space which are not embeddings and whose second fundamental form is semi-definite (see Spivak 1979, p. 124). Thus, in this context, to ensure that the immersion is actually an embedding, we need stronger conditions on the second fundamental form.

Currier (1989) obtained a Hadamard-Stoker type theorem for complete hypersurfaces in hyperbolic space which are locally supported by horospheres, that is, the eigenvalues of their shape operators are all greater than or equal to 1. Here, we consider the analogous problem in arbitrary codimension and obtain the following result.

Theorem 2 *Let $f: M^n \to \mathbb{H}^{n+p}$ be an isometric immersion of an orientable complete connected Riemannian manifold M^n into the hyperbolic space $\mathbb{H}^{n+p}$. Assume that there is an orthonormal frame $\{\xi_1, \dots, \xi_p\}$ in $TM^\perp$ such that all the eigenvalues of the shape operators A_{ξ_i} are greater than or equal to 1. Then, f admits a reduction of codimension to one, $f: M^n \to \mathbb{H}^{n+1}$. As a consequence, f is and embedding, $f(M)$ is the boundary of a convex set in $\mathbb{H}^{n+1}$, and M is either diffeomorphic to S^n or to $\mathbb{R}^n$. Moreover, f is rigid and, in fact, $f(M)$ is a horosphere of $\mathbb{H}^{n+1}$ if M is not compact.*

Essentially, the proof of Theorem 1 will be carried out by means of the so called Beltrami maps, which were used by do Carmo and Warner as well. These maps are geodesic diffeomorphisms from either an open hemisphere or the hyperbolic space to the Euclidean space of same dimension. By relying on their properties, one can reduce certain problems in spherical or hyperbolic spaces to analogous ones set in Euclidean space.

We also benefit from Beltrami maps to obtain an existence result for hypersurfaces in space forms with prescribed boundary and vanishing Gauss-Kronecker curvature (Corollary 1), as well as to establish two results regarding the convex hull of bounded domains of submanifolds of space forms (Corollaries 2 and 3). By the same token, we obtain the following extension of a classical theorem due to Hadamard (1898).

Theorem 3 *Let $f: M^n \to \mathcal{H}^{n+1}$ be a compact connected hypersurface, where $\mathcal{H}^{n+1}$ is either the open hemisphere of $S^{n+1} \subset \mathbb{R}^{n+2}$ centered at $e = (0, 0, \dots, 1)$ or the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2} = (\mathbb{R}^{n+2}, \langle\, , \,\rangle_{\mathbb{L}})$. Then, the following assertions are equivalent:*

(i) *The second fundamental form of f is definite everywhere.*
(ii) *The Gauss-Kronecker curvature of f is nowhere vanishing.*
(iii) *M is orientable and, for a unit normal field ξ defined on M, the map*

$$\begin{aligned} \psi : M^n &\to S^n \\ x &\mapsto \frac{\xi(x) - \langle \xi(x), e\rangle e}{\sqrt{1 - \langle \xi(x), e\rangle^2}} \end{aligned}$$

is a well-defined diffeomorphism, where S^n stands for the n-dimensional unit sphere of the Euclidean orthogonal complement of e in $\mathbb{R}^{n+2}$.

Furthermore, any of the above conditions implies that f is rigid and embeds M onto the boundary of a compact convex set in $\mathcal{H}^{n+1}$.

It is easily seen that the standard minimal immersion of the two-dimensional Clifford torus in S^3 has non-zero Gauss-Kronecker curvature, which shows that Theorem 3 is not valid for general compact hypersurfaces $f: M^n \to S^{n+1}$.

The second part of our paper is devoted to the problem of reducing the codimension of certain isometric immersions $f: M^n \to Q_c^{n+p}$, which we will call $(1\,;1)$-*semi-regular*. Such an immersion is characterized by the fact that, at each non-totally geodesic point, its first normal space (that is, the space generated by its second fundamental form) has constant dimension equal to 1. If f is $(1\,;1)$-semi-regular and has no totally geodesic points, it is called $(1\,;1)$-regular.

Our last result, as quoted below, extends the main results of Rodriguez and Tribuzy (1984) (theorems 1, 2, and 3) to the more general context of $(1\,;1)$-semi-regular isometric immersions. There, the authors studied reduction of codimension of $(1\,;1)$-regular isometric immersions into space forms.

Theorem 4 *Let M^n be a complete (compact, if $c \leq 0$) connected Riemannian manifold, and $f: M^n \to Q_c^{n+p}$ an $(1\,;1)$-semi-regular isometric immersion of M^n into the space form Q_c^{n+p}. Then, the immersion f admits a reduction of codimension to one whenever its set of totally geodesic points does not disconnect M. As a consequence, the following hold:*

(i) *f is an embedding and $f(M)$ bounds a compact convex set of Q_c^{n+1}, provided the Ricci curvature of M is nowhere less than c.*
(ii) *f is rigid if either $c > 0$ and $n \geq 4$ or $c \leq 0$ and $n \geq 3$.*

The paper is organized as follows. In Sect. 2, we introduce some notation and basic results on isometric immersions. In Sect. 3, we discuss on semi-regular isometric immersions and their elementary properties. In Sect. 4, we introduce the Beltrami maps and establish a result (Proposition 4) that will lead to the proofs of our theorems. In Sect. 5, we prove the theorems from 1 to 3, and finally, in Sect. 6, we prove Theorem 4.

2 Preliminaries

Let us fix some notation and recall some classical results on isometric immersions which will be used throughout the paper. For details and proofs we refer to Dajczer (1990).

Unless otherwise stated (*e.g.*, Corollary 1), all Riemannian manifolds we consider here are assumed to be C^∞ and of dimension $n \geq 2$. We shall denote by Q^n_c the n-dimensional space form whose sectional curvatures are constant and equal to $c \in \{0, 1, -1\}$, that is, Q^n_0 is the Euclidean space $\mathbb{R}^n$, Q^n_1 the unit sphere S^n, and Q^n_{-1} is the hyperbolic space $\mathbb{H}^n$.

Let M^n and $\tilde{M}^{n+p}$ be Riemannian manifolds of dimensions $n \geq 2$ and $n + p > n$, respectively. Given an isometric immersion

$$f : M^n \to \tilde{M}^{n+p},$$

we will write TM and $TM^\perp$ for its tangent bundle and normal bundle, respectively, and $\alpha_f : TM \times TM \to TM^\perp$ for its second fundamental form, that is,

$$\alpha_f(X, Y) = \widetilde{\nabla}_X Y - \nabla_X Y,$$

where $\widetilde{\nabla}$ and ∇ denote, respectively, the Riemannian connections of $\tilde{M}$ and M.

Given $\xi \in TM^\perp$, we define the (self-adjoint) operator $A_\xi : TM \to TM$ by

$$A_\xi X = -(\text{tangential component of } \widetilde{\nabla}_X \xi)$$

and call it the *shape operator* of f in the normal direction ξ. It is easily seen that

$$\langle A_\xi X, Y\rangle = \langle \alpha_f(X, Y), \xi\rangle \quad \forall X, Y \in TM, \;\; \xi \in TM^\perp,$$

where $\langle\ ,\ \rangle$ stands for the Riemannian metric in both M and $\tilde{M}$.

One says that the second fundamental form α_f of f is *semi-definite* if, for all $\xi \in TM^\perp$, the 2-form

$$(X, Y) \in TM \times TM \mapsto \langle \alpha_f(X, Y), \xi\rangle$$

is semi-definite. Clearly, this condition is equivalent to that of all the nonzero eigenvalues of the shape operator A_ξ having the same sign.

The set of totally geodesic points of an isometric immersion $f : M^n \to \tilde{M}^{n+p}$, that is, those at which the second fundamental form vanishes, will be denoted by M_{tot}. If, in addition, $\tilde{M}$ has constant curvature c, we will denote by M_c the set of points x of M whose sectional curvatures $K_M(X, Y)$, $X, Y \in T_x M$, are all equal to c. Observe that, by *Gauss equation*

$$K_M(X, Y) = c + \langle \alpha_f(X, X), \alpha_f(Y, Y)\rangle - \|\alpha_f(X, Y)\|^2,$$

which is valid for any orthonormal vectors $X, Y \in TM$, we have that

$$M_{\rm tot} \subset M_c \subset M.$$

We define the *subspace of relative nullity* of f at $x \in M$ as the vector subspace $\Delta(x)$ of T_xM given by

$$\Delta(x) = \{X \in T_xM\,;\ \alpha_f(X,Y) = 0\,\forall Y \in T_xM\}.$$

The dimension of $\Delta(x)$ is called the *index of relative nullity* of f at x, and is denoted by $\nu(x)$. We also define the *index of minimum relative nullity* of f by

$$\nu_{\min} = \min_{x\in M} \nu(x).$$

Finally, let us recall that an isometric immersion $f\colon M^n \to Q_c^{n+p}$ is called *rigid* if, for any isometric immersion $g\colon M^n \to Q_c^{n+p}$, there is an ambient isometry $\Phi\colon Q_c^{n+p} \to Q_c^{n+p}$ such that $g = \Phi \circ f$.

Most of our results on rigidity here will follow from the following theorem, due to Sacksteder (1962).

Sacksteder Rigidity Theorem *Let $f\colon M^n \to Q_c^{n+1}$ be an isometric immersion of a compact (resp. complete) Riemannian manifold with $n \geq 3$ and $c \leq 0$ (resp. $n \geq 4$ and $c > 0$). Then, f is rigid, provided its set of totally geodesic points does not disconnect M.*

3 Semi-Regular Isometric Immersions

We say that an isometric immersion $f\colon M^n \to Q_c^{n+p}$ admits a *reduction of codimension* to $q < p$, if there is a totally geodesic submanifold $Q_c^{n+q} \subset Q_c^{n+p}$ such that $f(M) \subset Q_c^{n+q}$. The immersion f is also said to be $(1\,; q)$-*regular* if, for all $x \in M$, the *first normal space* of f at x,

$$\mathscr{N}(x) := \text{span}\{\alpha_f(X,Y)\,;\ X, Y \in T_xM\},$$

has constant dimension q. In this case,

$$\mathscr{N} := \bigcup_{x\in M} \mathscr{N}(x)$$

is a subbundle of $TM^\perp$ which will be called the *first normal bundle* of f. $\mathscr{N}$ is said to be *parallel* if, at any point $x \in M$, one has

$$\nabla_X^\perp \xi \in \mathscr{N}(x) \quad \forall\, X \in T_xM,\ \xi \in \mathscr{N}(x),$$

where $\nabla^\perp$ stands for the normal connection of f.

For future reference, let us quote a standard result on reduction of codimension of isometric immersions.

Proposition 1 *Let $f\colon M^n \to Q_c^{n+p}$ be a connected $(1\,;q)$-regular isometric immersion whose first normal bundle $\mathscr{N}$ is parallel. Then, f admits a reduction of codimension to q.*

Proof See Corollary 4.2 of Dajczer (1990). □

For our purposes, it will be convenient to introduce the following concept.

Definition 1 An isometric immersion $f\colon M^n \to Q_c^{n+p}$ will be called $(1\,;q)$-*semi-regular*, if it is non-totally geodesic and its restriction to $M - M_{\rm tot}$ is $(1\,;q)$-regular.

We establish now two elementary results regarding semi-regular isometric immersions. The first one appears in Rodriguez and Tribuzy (1984) (Lemma 2). For the reader's convenience, we will present it here with a slightly different proof.

First, recall the *Codazzi equation* for isometric immersions $f\colon M^n \to Q_c^{n+p}$:

$$\left(\nabla_X^{\perp}\alpha_f\right)(Y,Z) = \left(\nabla_Y^{\perp}\alpha_f\right)(X,Z), \quad X,Y,Z \in TM,$$

where

$$\left(\nabla_X^{\perp}\alpha_f\right)(Y,Z) := \nabla_X^{\perp}\alpha_f(Y,Z) - \alpha_f\left(\nabla_X Y, Z\right) - \alpha_f\left(Y, \nabla_X Z\right).$$

Proposition 2 *Let $f\colon M^n \to Q_c^{n+p}$ be an $(1\,;1)$-semi-regular isometric immersion such that $M - M_c$ is nonempty. Then, $\mathscr{N}$ is parallel in $M - M_c$.*

Proof Given $x \in M - M_c$, let ξ be a unit normal field which spans $\mathscr{N}$ in an open neighborhood V of x in $M - M_c$, and $\{X_1,\dots,X_n\}$ an orthonormal frame in TV that diagonalizes A_ξ with corresponding eigenvalues $\lambda_1,\dots,\lambda_n$. In this setting, one has

$$\alpha_f(X_i,X_j) = \langle\alpha_f(X_i,X_j),\xi\rangle\xi = \langle A_\xi X_i, X_j\rangle\xi = \delta_{ij}\lambda_i\xi. \tag{1}$$

By Gauss equation, for all $i \neq j \in \{1,\dots,n\}$,

$$K_M(X_i,X_j) = c + \lambda_i\lambda_j\,. \tag{2}$$

Hence, at each point of $V \subset M - M_c$, A_ξ has at least two nonzero eigenvalues, that is, for a given $i \in \{1,\dots,n\}$, there exists $j \neq i$, such that $\lambda_j \neq 0$. Moreover, from (1), we have that $\alpha_f(X_i,X_j) = 0$ and $\alpha_f(X_j,X_j) = \lambda_j\xi$. Thus, the following equalities hold:

- $\left(\nabla_{X_j}^{\perp}\alpha_f\right)(X_i,X_j) = -\alpha_f\left(\nabla_{X_j}X_i, X_j\right) - \alpha_f\left(X_i, \nabla_{X_j}X_j\right).$
- $\left(\nabla_{X_i}^{\perp}\alpha_f\right)(X_j,X_j) = (X_i\lambda_j)\xi + \lambda_j\nabla_{X_i}^{\perp}\xi - 2\alpha_f\left(\nabla_{X_i}X_j, X_j\right).$

Now, the Codazzi equation gives that the right hand sides of these two equalities coincide, which implies that $\nabla^{\perp}_{X_i}\xi(x) \in \mathscr{N}(x)$. Therefore, the first normal bundle $\mathscr{N}$ is parallel in $M - M_c$. □

Proposition 3 *Let $f : M^n \to Q_c^{n+p}$ be a non-totally geodesic isometric immersion with semi-definite second fundamental form, and H its mean curvature field. Then, the following hold:*

(i) *$x \in M_{\text{tot}}$ if and only if $H(x) = 0$.*
(ii) *For $x \in M - M_{\text{tot}}$, $\mathscr{N}(x) = \text{span}\,\{H(x)\}$ and all the eigenvalues of A_H are nonnegative. In particular, f is $(1\,;1)$-semi-regular.*
(iii) *$K_M \geq c$ everywhere.*

Proof By definition, one has

$$H = \sum_{i=1}^{n} \alpha_f(X_i\,, X_i) = \sum_{j=1}^{p} (\text{trace}\, A_{\xi_j})\xi_j\,, \tag{3}$$

where $\{X_1\,, \ldots, X_n\}$ and $\{\xi_1\,, \ldots, \xi_p\}$ are arbitrary orthonormal frames in TM and $TM^{\perp}$, respectively.

Since f has semi-definite second fundamental form, a shape operator A_ξ of f is identically zero if and only if its trace is equal to zero. Consequently, f is totally geodesic at $x \in M$ if and only if $H(x) = 0$, which proves (i).

For $x \in M - M_{\text{tot}}$, we can choose an orthonormal frame $\{\xi_1\,, \ldots, \xi_p\} \subset TM^{\perp}$ in an open neighborhood U of x, with $\xi_1 = H/\|H\|$. In this case, (3) yields

$$\text{trace}\, A_{\xi_1} = \|H\| > 0 \quad \text{and} \quad \text{trace}\, A_{\xi_j} = 0 \text{ (i.e. } A_{\xi_j} = 0) \;\forall j = 2, \ldots, p.$$

Therefore, for $X, Y \in TU$, one has

$$\alpha_f(X, Y) = \sum_{j=1}^{p} \langle \alpha_f(X, Y), \xi_j \rangle \xi_j = \sum_{j=1}^{p} \langle A_{\xi_j} X, Y \rangle \xi_j = \langle A_{\xi_1} X, Y \rangle \xi_1\,,$$

which implies that $\mathscr{N}(x) = \text{span}\,\{H(x)\}$. Since the trace of A_{ξ_1} is positive, we also have that all of its eigenvalues are nonnegative. This proves (ii).

Now, considering an orthonormal basis of T_xM that diagonalizes A_H and the Gauss equation (as in (2)), we easily conclude that $K_M \geq c$ at x. Since $K_M = c$ on M_{tot}, we have that $K_M \geq c$ on all of M, which proves (iii). □

Remark 1 By means of the Gauss equation, as in the last paragraph of the above proof, we conclude that for $x \in M_c - M_{\text{tot}}$, there is exactly one nonzero eigenvalue of A_H. So, *for any* $(1\,;1)$-*semi-regular isometric immersion* $f : M^n \to Q_c^{n+p}$,

$$\nu(x) = n - 1 \;\; \forall x \in M_c - M_{\text{tot}}\,.$$

4 Beltrami Maps

Given a point e in $S^n \subset \mathbb{R}^{n+1}$, let $\mathcal{H}_e$ be the open hemisphere of S^n centered at e, that is, the open geodesic ball of S^n with center at e and radius $\pi/2$. The central projection φ from $\mathcal{H}_e$ to the tangent space of S^n at e, which we identify with $\mathbb{R}^n$, is a diffeomorphism called the *Beltrami map* of $\mathcal{H}_e$.

For $e = (0, \dots, 0, 1)$, this map is defined as

$$\begin{array}{rcl} \varphi : \mathcal{H}_e & \to & \mathbb{R}^n \\ x & \mapsto & \frac{x}{x_{n+1}} \end{array}, \tag{4}$$

where x_{n+1} stands for the $(n+1)$-th coordinate of x in $\mathbb{R}^{n+1}$.

It is easily seen that φ and its inverse are both geodesic maps, that is, they take geodesics to geodesics and, in particular, convex sets to convex sets.

Now, consider the Lorentzian space $\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_{\mathbb{L}})$, where

$$\langle x\,, x\rangle_{\mathbb{L}} = \sum_{i=1}^{n} x_i^2 - x_{n+1}^2\,,\ \ x = (x_1\,, \dots, x_{n+1}),$$

and the hyperboloid model of hyperbolic space

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1};\ \langle x\,, x\rangle_{\mathbb{L}} = -1 \ \text{ and } \ x_{n+1} > 0\}.$$

Let us identify the affine subspace $x_{n+1} = 1$ of $\mathbb{L}^{n+1}$ with $\mathbb{R}^n$. Denoting its unit open ball centered at 0 by B^n, and considering in it the Euclidean metric, one has that the *Beltrami map* of $\mathbb{H}^n$,

$$\begin{array}{rcl} \varphi : \mathbb{H}^n & \to & B^n \\ x & \mapsto & \frac{x}{x_{n+1}} \end{array}, \tag{5}$$

is also a geodesic diffeomorphism. This follows from the well known fact that, endowed with a suitable metric, B^n represents the Kleinian model of hyperbolic space. In this setting, the map φ becomes an isometry (in particular, a geodesic diffeomorphism) between the Lorentzian and the Kleinian models of $\mathbb{H}^n$. However, the geodesics of the Kleinian model are precisely the segments of Euclidean lines which lie in B^n.

The following result, which generalizes Proposition 2.3 of do Carmo and Warner (1970), will play a fundamental role in the proofs of our theorems.

Proposition 4 *Let $\mathcal{H}^{n+p}$ be either an open hemisphere of S^{n+p} or the hyperbolic space $\mathbb{H}^{n+p}$, and $\varphi : \mathcal{H}^{n+p} \to \mathbb{R}^{n+p}$ its corresponding Beltrami map. Suppose that $f : M^n \to \mathcal{H}^{n+p}$ is an isometric immersion with second fundamental form α_f. Consider also in M the metric induced by the immersion*

$$\bar{f} := \varphi \circ f : M^n \to \mathbb{R}^{n+p}$$

and denote its second fundamental form by $\alpha_{\bar{f}}$. Under these conditions, the following hold:

(i) *At a point* $x \in M$, α_f *is semi-definite (resp. definite) if and only if* $\alpha_{\bar{f}}$ *is semi-definite (resp. definite).*
(ii) *The set of totally geodesic points of* f *and* $\bar{f}$ *coincide.*

Proof In the spherical case, we may assume without loss of generality that $\mathcal{H}^{n+p}$ is the open hemisphere of S^{n+p} centered at $e = (0, \dots, 0, 1) \in \mathbb{R}^{n+p+1}$. In this way, we can write

$$\varphi(x) = \phi(x)x, \quad \phi(x) = \frac{1}{x_{n+p+1}}, \quad x = (x_1, \dots, x_{n+p+1}) \in \mathcal{H}^{n+p}, \tag{6}$$

and treat both the spherical and hyperbolic cases simultaneously.

Recall that the Riemannian connection of $\mathbb{R}^{n+p+1}$, which we will denote by $\widehat{\nabla}$, is the same for both the Euclidean and Lorentzian metrics. So, denoting the Riemannian connection of $\mathcal{H}^{n+p}$ by $\widetilde{\nabla}$, one has

$$\widehat{\nabla}_X Y = \widetilde{\nabla}_X Y - \langle X, Y \rangle I \quad \forall X, Y \in T\mathcal{H}^{n+p}, \tag{7}$$

where I stands for the identity field of $\mathbb{R}^{n+p+1}$ and $\langle\, , \rangle$ is either the Euclidean or Lorentzian metric of $\mathbb{R}^{n+p+1}$.

Denote by $\overline{M}$ the manifold M endowed with the metric induced by $\bar{f}$. Considering (6), we have for all $X \in TM$ that

$$\overline{X} := \varphi_* X = (X\phi)I + \phi X \in T\overline{M} \subset \Pi, \tag{8}$$

where Π denotes the Euclidean orthogonal complement of e in $\mathbb{R}^{n+p+1}$.

Now, given $\xi \in TM^\perp \subset T\mathcal{H}^{n+p}$, one has $\langle \xi, I \rangle = 0$. Thus, writing $\bar{\xi}$ for the component of ξ in Π, that is,

$$\bar{\xi} := \xi - (\xi_{n+p+1})e, \quad \xi \in TM^\perp, \tag{9}$$

we have that $\bar{\xi} \in T\overline{M}^\perp \subset \Pi$. Indeed, for all $\overline{X} = \varphi_* X \in T\overline{M}$, one has

$$\langle \bar{\xi}, \overline{X} \rangle = \langle \xi, \overline{X} \rangle = \langle \xi, (X\phi)I + \phi X \rangle = 0.$$

Let us fix a point $x \in M$. Since φ is a diffeomorphism and no tangent space to $\mathcal{H}^{n+p}$ is orthogonal to Π, for a suitable open neighborhood U of x, the correspondences defined in (8) and (9),

$$X \in TU \mapsto \overline{X} \in T\overline{U} \quad \text{and} \quad \xi \in TU^\perp \mapsto \bar{\xi} \in T\overline{U}^\perp,$$

are bundle isomorphisms.

Now, extending a given $\xi \in TU^\perp$ by parallel displacement along the lines through the origin of $\mathbb{R}^{n+p+1}$, we have that $\widehat{\nabla}_I \xi = 0$. Thus,

$$\widehat{\nabla}_{\overline{X}} \xi = \phi \widehat{\nabla}_X \xi \quad \forall\, \overline{X} = \varphi_* X \in T\overline{U}. \tag{10}$$

Furthermore, given $X \in TU$, one has

$$0 = X\langle \xi, I\rangle = \langle \widehat{\nabla}_X \xi, I\rangle + \langle \xi, \widehat{\nabla}_X I\rangle = \langle \widehat{\nabla}_X \xi, I\rangle + \langle \xi, X\rangle = \langle \widehat{\nabla}_X \xi, I\rangle \tag{11}$$

and, from (7), that $\widehat{\nabla}_X \xi = \widetilde{\nabla}_X \xi$. This, together with (10) and (11), yields

$$\langle \alpha_{\bar f}(\overline{X}, \overline{X}), \bar\xi\rangle = -\langle \widehat{\nabla}_{\overline{X}} \bar\xi, \overline{X}\rangle = -\langle \widehat{\nabla}_{\overline{X}} \xi, \overline{X}\rangle = -\phi^2\langle \widehat{\nabla}_X \xi, X\rangle = -\phi^2\langle \widetilde{\nabla}_X \xi, X\rangle,$$

which implies that the equality

$$\langle \alpha_{\bar f}(\overline{X}, \overline{X}), \bar\xi\rangle = \phi^2\langle \alpha_f(X, X), \xi\rangle \tag{12}$$

holds everywhere in U and, in particular, at x. Since the function ϕ never vanishes, the assertions (i) and (ii) easily follow. □

Remark 2 In the above proof, equation (12) gives that the ranks of the shape operators A_ξ and $A_{\bar\xi}$ coincide. Consequently, when the codimension p is 1 and M is oriented, $x \in M$ is a point of vanishing Gauss-Kronecker curvature for f if and only if it is for $\bar f = \varphi \circ f$ (recall that the *Gauss-Kronecker curvature* of an oriented hypersurface $f: M^n \to Q_c^{n+1}$ is the determinant of its shape operator).

By combining Proposition 4 with an outstanding result by Guan and Spruck (2002), we obtain the following existing result for hypersurfaces with prescribed boundary and vanishing Gauss-Kronecker curvature in space forms.

Corollary 1 *Let $\Gamma^{n-1} \subset \mathcal{H}^{n+1}$ be a compact embedded (not necessarily connected) $(n-1)$-dimensional submanifold of $\mathcal{H}^{n+1}$. Assume that there exists a connected compact oriented C^2 hypersurface with boundary $g: N^n \to \mathcal{H}^{n+1}$, satisfying $g(\partial N) = \Gamma$, whose second fundamental form is semi-definite on N and definite in a neighborhood of ∂N. Under these conditions, there exists a connected compact oriented $C^{1,1}$ (up to the boundary) hypersurface $f: M^n \to \mathcal{H}^{n+1}$ with semi-definite second fundamental form, which satisfies $f(\partial M) = \Gamma$ and has vanishing Gauss-Kronecker curvature everywhere.*

Proof It follows from Proposition 4 that the second fundamental form of the immersion $\bar g = \varphi \circ g: N \to \mathbb{R}^{n+1}$ is semi-definite, and definite in a neighborhood of ∂N. Thus, $\overline{\Gamma} := \varphi(\Gamma) = \bar g(\partial N)$ is a compact embedded submanifold of $\mathbb{R}^{n+1}$ which fulfills the hypothesis of Theorem 1.2 in Guan and Spruck (2002). Therefore, there exists a connected compact oriented $C^{1,1}$ (up to the boundary) hypersurface with boundary, $\bar f: M \to \mathbb{R}^{n+1}$, whose second fundamental form is semi-definite, which satisfies $\bar f(\partial M) = \overline{\Gamma}$ and has vanishing Gauss-Kronecker curvature everywhere. Hence (see Remark 2), $f = \varphi^{-1} \circ \bar f : M^n \to \mathcal{H}^{n+1}$ is the desired hypersurface. □

An isometric immersion $f: M^n \to \tilde M^{n+p}$ is said to have the *convex hull property* if, for every domain D on M such that $f(D)$ is bounded in $\tilde M$, $f(D)$ lies in the convex hull of its boundary in $\tilde M$. It is a well known fact that minimal immersions in Euclidean space have this property.

In [15], Osserman established that an isometric immersion into Euclidean space has the convex hull property if and only if there is no normal direction in which its second fundamental form is definite. Since Beltrami maps are convexity-preserving, Osserman's theorem and Proposition 4 give the following result, which, in the spherical case, generalizes a theorem due to Lawson (1970, Theorem 1').

Corollary 2 *An isometric immersion* $f\colon M^n \to \mathcal{H}^{n+p}$ *has the convex hull property if and only if there is no normal direction* ξ *for which the* 2*-form*

$$(X, Y) \in TM \times TM \mapsto \langle \alpha_f(X, Y), \xi\rangle$$

is definite. In particular, any minimal isometric immersion $f\colon M^n \to \mathcal{H}^{n+p}$ *has the convex hull property.*

In Alexander and Ghomi (2003), Alexander and Ghomi showed that, in Euclidean space, the convex hull of the boundary of a hypersurface whose second fundamental form is semi-definite has a property which is a dual of the classical one we considered above. This result, together with Proposition 4, yields the following

Corollary 3 *Let* $f\colon M^n \to \mathcal{H}^{n+1}$ *be a compact connected hypersurface with boundary, whose second fundamental form* α_f *is semi-definite. Let* C *be the convex hull of* $f(\partial M)$ *and assume that:*

(i) $f(\partial M) \subset \partial C$.
(ii) α_f *is definite in a neighborhood of* ∂M.
(iii) f *is an embedding on each component of* ∂M.

Then, the image of the interior of M *lies completely outside* C*, that is,*

$$f(\operatorname{int} M) \cap C = \emptyset.$$

We remark that, in Corollary 3, none of the conditions (i)—(iii) can be omitted (see Alexander and Ghomi 2003).

5 Proofs of Theorems 1–3

We proceed now to the proofs of the theorems. To show Theorem 1, we recover the Euclidean case by means of Proposition 4, and then apply a theorem by Jonker (1975), stated below. The same goes to Theorem 3, in which a result due to Hadamard (1898) plays the role of Jonker's Theorem. Theorem 2 will be derived from propositions 1 and 2, Theorem 1, and the main results of Currier (1989).

Jonker's Theorem *Let* M^n *be a complete connected Riemannian manifold, and* $f\colon M^n \to \mathbb{R}^{n+p}$ *an isometric immersion with semi-definite second fundamental form. Then, one of the following two possibilities holds:*

(i) *There is an affine subspace* $\mathbb{R}^{n+1}$ *of* $\mathbb{R}^{n+p}$ *such that* f *embeds* M *onto the boundary of a convex set of* $\mathbb{R}^{n+1}$*. In particular,* M *is diffeomorphic to* S^n *or to* $\mathbb{R}^n$.

(ii) *The immersion* f *is an* $(n-1)$*-cylinder over a curve* $\gamma : \mathbb{R} \to \mathbb{R}^{p+1}$, *that is,* M *and* f *split respectively as* $\mathbb{R} \times \mathbb{R}^{n-1}$ *and* $\gamma \times \mathrm{id}$, *where* id *stands for the identity map of* $\mathbb{R}^{n-1}$.

Remark 3 Along the proof of Jonker's Theorem, it is shown that the possibility (i) occurs only if $M - M_{\mathrm{tot}}$ is connected.

Proof of Theorem 1 Let us prove first that, for $c = 1$, there is an open hemisphere of S^{n+p} which contains $f(M)$.

Since α_f is semi-definite, by Proposition 3, the first normal space of f at any point $x \in M - M_{\mathrm{tot}}$ is spanned by the mean curvature vector $H(x)$. In this case, there is a point $x \in M - M_{\mathrm{tot}}$ at which α_f is positive-definite in the direction $H(x)$, that is, all the eigenvalues of A_H at x are positive. Indeed, if it were not so, the index of minimum relative nullity of f, $\nu_{\min}$, would satisfy $0 < \nu_{\min} < n$. Then, by a result due to Dajczer and Gromoll (1985), at any $x \in M$ such that $\nu(x) = \nu_{\min}$, the number of positive and negative eigenvalues of A_H would be equal. However, by Proposition 3, all the eigenvalues of A_H are nonnegative.

Let then $x \in M - M_{\mathrm{tot}}$ be a point at which α_f is positive-definite in the direction $H(x)$. Denote by $\mathcal{H}_{f(x)}$ the open hemisphere of S^{n+p} centered at $f(x)$, and let $\varphi : \mathcal{H}_{f(x)} \to \mathbb{R}^{n+p}$ be its Beltrami map. Write N for the connected component of $f^{-1}(\mathcal{H}_{f(x)})$ that contains x, and endow it with the metric induced by the immersion $\bar{f} = \varphi \circ f|_N : N \to \mathbb{R}^{n+p}$. By Proposition 4, $\bar{f}$ is an isometric immersion with semi-definite second fundamental form. Furthermore, as shown in Lemma 2.5 of do Carmo and Warner (1970), which is valid regardless of the codimension, $\bar{f}$ is also complete.

It follows from these considerations and Jonker's Theorem that $\bar{f}(N) \subset \mathbb{R}^{n+p}$ is either a cylinder over a curve or the boundary of a convex set in an $(n+1)$-dimensional affine subspace of $\mathbb{R}^{n+p}$, which we identify with $\mathbb{R}^{n+1}$. In the latter case, $\bar{f}|_N : N \to \mathbb{R}^{n+1}$ is an embedding. However, by Proposition 4, $\alpha_{\bar{f}}$ is definite at x, which excludes the possibility of $\bar{f}(N)$ being a cylinder.

The definiteness of $\alpha_{\bar{f}}$ at x also implies that there exists an open neighborhood U of x in M, such that $\bar{f}(\mathrm{cl}(U) - \{x\})$ is contained in one of the open semi-spaces of $\mathbb{R}^{n+1}$ determined by $\Sigma := \bar{f}_*(T_x M)$, say Σ_+ (see Fig. 1).

Let $\eta \in \mathbb{R}^{n+1}$ be the unit normal to $\bar{f}(M)$ at $\bar{f}(x)$ pointing to Σ_+. If we write Π for the orthogonal complement of η in $\mathbb{R}^{n+p}$, we have that $\bar{f}(\mathrm{cl}(U) - \{x\})$ is contained in the open semi-space Π_+ of $\mathbb{R}^{n+p}$ which is determined by Π and contains Σ_+. Hence, denoting by $\mathcal{H}$ the open hemisphere of S^{n+p} that contains $\varphi^{-1}(\Pi_+)$, we have that $f(x) \in \partial\mathcal{H}$ and $f(\mathrm{cl}\, U - \{x\}) \subset \mathcal{H}$.

Now, notice that $\partial\mathcal{H}$ is a local $(n+p-1)$-dimensional "supporting sphere" for $f(M)$ at $f(x)$. This fact will allow us to adapt the reasoning of the last two paragraphs of the proof of Theorem 1.1(a) in do Carmo and Warner (1970) to conclude that, in fact, $f(M) - \{f(x)\} \subset \mathcal{H}$.

The argument goes as follows. Let $C \subset f^{-1}(\mathcal{H}) \subset M$ be the connected component of $f^{-1}(\mathcal{H})$ that contains $\mathrm{cl}\, U - \{x\}$. By abuse of notation, write φ for the Beltrami map of $\mathcal{H}$ and set $\bar{f} = \varphi \circ f : C \to \mathbb{R}^{n+p}$. Thus, $\bar{f}$ is a noncompact complete and non-flat isometric immersion with semi-definite second fundamental form.

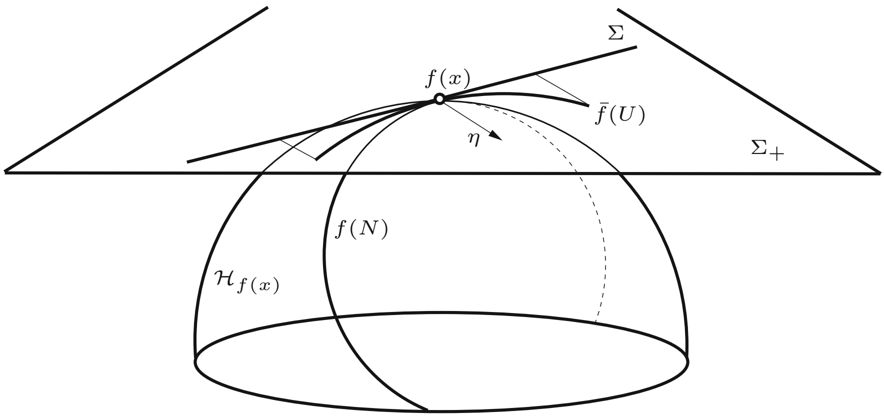

Fig. 1 The hemisphere determined by $f(x)$ and its Beltrami map

By Jonker's Theorem, $\bar{f}(C)$ is diffeomorphic to $\mathbb{R}^n$. Assume that ∂U is a geodesic sphere of M centered at x so that $\bar{f}(\partial U)$ separates $\bar{f}(C)$ into two connected components, being one of them bounded. Denoting by Ω this bounded component, we have that $\bar{f}(C - \mathrm{cl}\, U) \subset \Omega$. Otherwise, there would be $y \in C - \mathrm{cl}\, U$ such that $\bar{f}(y) \in \bar{f}(C) - \mathrm{cl}\, \Omega$. Then, a minimal geodesic γ joining y to x would cross ∂U only once, and its image $\bar{f}(\gamma)$ would be unbounded. This, however, is a contradiction, for $\bar{f}(\gamma)$ comes from the unbounded component $f(C) - \mathrm{cl}\, \Omega$, and cross $\partial\Omega$ only once. Therefore, $f(x)$ is the only limit point of $f(C)$ on $\partial\mathcal{H}$, that is, $f(C) \subset f(\mathrm{cl}\, U - \{x\}) \subset \mathcal{H}$. Thus, since M is connected, one has $f(M) - \{f(x)\} \subset \mathcal{H}$, as claimed.

Finally, choosing $e \in S^{n+p}$ sufficiently close to the center of $\mathcal{H}$, we have that $f(M)$ is contained in the open hemisphere of S^{n+p} centered at e, as we wished to prove.

Suppose now that $c \neq 0$ and let $\mathcal{H}^{n+p}$ be either an open hemisphere of S^{n+p} that contains $f(M)$ or the hyperbolic space $\mathbb{H}^{n+p}$. As before, if we write $\varphi : \mathcal{H}^{n+p} \to \mathbb{R}^{n+p}$ for the Beltrami map of $\mathcal{H}^{n+p}$, we have that the second fundamental form of the immersion $\bar{f} = \varphi \circ f$ is semi-definite. Thus, since M is compact, it follows from Jonker's Theorem that $\bar{f}$ is an embedding and $\bar{f}(M)$ is the boundary of a compact convex set in an $(n+1)$-dimensional affine subspace $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+p}$. Therefore, $f = \varphi^{-1} \circ \bar{f}$ embeds M onto the boundary of a compact convex set of $\mathcal{H}^{n+1} = \varphi^{-1}(\mathbb{R}^{n+1}) \subset \mathcal{H}^{n+p}$, for φ is convexity-preserving.

Since we have reduced the codimension to one, we can apply the main result of do Carmo and Warner (1970) to conclude that f is rigid for $c = 1$. For $c = -1$ and $n = 2$, the rigidity of f follows from Theorem 5 of Fomenko and Gajubov (1972), which, in fact, is set in the more general context of compact surfaces with boundary.

Suppose then that $c = -1$ and $n > 2$. Since the set of totally geodesic points of $\bar{f}$ does not disconnect M (see Remark 3), it follows from Proposition 4-(ii) that the same is true for f. Thus, Sacksteder Rigidity Theorem applies and gives that f is rigid. This concludes our proof. □

Proof of Theorem 2 It follows from theorems A and B of Currier (1989) that, except for the rigidity in the compact case, the result is true if the codimension of f is equal to one. Otherwise, we can reduce the codimension to one.

Indeed, since α_f is clearly positive-definite, we have that M_c is empty and, by Proposition 3, that f is $(1\,;\,1)$-regular. So, by Proposition 2, the first normal bundle of f is parallel. Since M is connected, Proposition 1 gives that f admits a reduction of codimension to one.

The rigidity of f when M is compact follows from Theorem 1. □

Proof of Theorem 3 Considering Proposition 4 together with its notation, we have that α_f is definite if and only if $\alpha_{\bar{f}}$ is definite. But, by the main result in Hadamard (1898) (see also Dajczer 1990, Chapter 2), $\alpha_{\bar{f}}$ is definite if and only if $\bar{f}$, and so f, has non-vanishing Gauss-Kronecker curvature (see Remark 2). This proves the equivalence between (i) and (ii).

Now, notice that $\psi = \bar{\xi}/\|\bar{\xi}\|$ is nothing but the Gauss map of $\bar{f}$. So, again by Hadamard Theorem, M is orientable and ψ is a diffeomorphism if and only if $\alpha_{\bar{f}}$, and so α_f, is definite, which shows that (i) is also equivalent to (iii).

The last assertion follows directly from Theorem 1. □

6 Proof of Theorem 4

The proof of Theorem 4 will be based on some properties of the relative nullity distribution that we shall introduce in the following.

Let $f : M^n \to \tilde{Q}_c^{n+p}$ be an isometric immersion whose index of relative nullity is constant and equal to $\nu_0 > 0$ in some open set $U \subset M$. It is well known that, under these conditions, the corresponding relative nullity distribution

$$x \in U \mapsto \Delta(x) \subset T_x M$$

is smooth and integrable, and its leaves are totally geodesic in M and Q_c^{n+p}. Furthermore, if $\nu_0 = \nu_{\min}$, these leaves are complete whenever M is complete.

In this setting, assuming that $\gamma : [0, a] \to M$ is a geodesic of M such that $\gamma([0, a))$ is contained in a leaf $\mathfrak{L} \subset U$ of the relative nullity distribution Δ, we have that

(P1) $\nu(\gamma(a)) = \nu_0$.

(P2) The first normal space $\mathscr{N}$ is parallel at $\gamma(t)\ \forall t \in [0, a)$, provided $\mathscr{N}$ is parallel at $\gamma(a)$.

Property **(P1)**, together with the integrability of the relative nullity distribution, is proved in Ferus (1971) (see also Dajczer 1990–Chapter 5), and property **(P2)** is the content of Lemma 4 of Rodriguez and Tribuzy (1984).

The next result is due to D. Ferus, which he used in his proof of Sacksteder Rigidity Theorem, as presented in Dajczer (1990)–Chapter 6.

Proposition 5 (Ferus) *Let $f : M^n \to Q_c^{n+p}$ be an isometric immersion, where M^n is assumed to be compact for $c \le 0$, and complete otherwise. Assume further that, in an open set $U \subset M$, $\nu = n - 1$. Then, no leaf of the relative nullity foliation is complete.*

Proof of Theorem 4 Let us notice first that $M - M_c$ is nonempty. Indeed, if it were not so, we would have $\nu = n - 1$ in the open set $M_c - M_{\rm tot} = M - M_{\rm tot}$ (see Remark 1). In this case, $\Delta \subset T(M - M_{\rm tot})$ would be the minimum relative nullity distribution and its leaves would then be complete, since M is complete. But, by Proposition 5, this is impossible.

Next, we prove that the first normal bundle $\mathscr{N}$ of f is parallel in $M - M_{\rm tot}$. From Proposition 2, $\mathscr{N}$ is parallel in the closure of $M - M_c$. So, it remains to prove that $\mathscr{N}$ is parallel in $U = {\rm int}(M_c - M_{\rm tot})$, which we can assume is nonempty. Given $x \in U$, it follows from Proposition 5 that any geodesic contained in a leaf $\mathfrak{L} \ni x$, with x as initial point, must intersect the boundary of U at a point y. However, by property **(P1)**, $\nu(y) = n - 1$. Hence, $y \notin M_{\rm tot}$, which implies that y is in the closure of $M - M_c$. In particular, $\mathscr{N}$ is parallel at y. This, together with property **(P2)**, gives that $\mathscr{N}$ is parallel at x, and thus in all of $M - M_{\rm tot}$.

Since we are assuming that $M - M_{\rm tot}$ is connected, it follows from Proposition 1 that $f({\rm cl}(M - M_{\rm tot}))$ is contained in a totally geodesic submanifold Q_c^{n+1} of Q_c^{n+p}. In particular, for all $x \in {\rm cl}(M - M_{\rm tot})$, one has $f_*(T_x M) \subset T_{f(x)} Q_c^{n+1}$.

If int $M_{\rm tot}$ is nonempty, according to results in Obata (1968), the generalized Gauss map of f is constant on any connected component C of $M_{\rm tot}$, which implies that $f(C)$ is contained in an n-dimensional totally geodesic submanifold $Q_c^n \subset Q_c^{n+p}$. So, by continuity, for any $x \in \partial C$, one has

$$T_{f(x)} Q_c^n = f_*(T_x M) \subset T_{f(x)} Q_c^{n+1},$$

that is, Q_c^n is tangent to Q_c^{n+1} at $f(x)$. Therefore, $f(C) \subset Q_c^n \subset Q_c^{n+1}$, from which we infer that $f(M_{\rm tot}) \subset Q_c^{n+1}$, and thus that $f(M) \subset Q_c^{n+1}$.

Now, since we have reduced the codimension to one, statement (ii) follows directly from the hypothesis and Sacksteder Rigidity Theorem.

Finally, in order to prove (i), assume that ${\rm Ric}_M \geq c$. Then, by the Bonnet-Myers Theorem, M is compact for $c > 0$.

Given a unit tangent field $X \in TM$, if $\{X_1, X_2, \ldots, X_n\} \in TM$ is an orthonormal tangent frame on M such that $X_1 = X$, it is well known that the equality

$${\rm Ric}_M(X) = c + \frac{1}{n-1}\langle \alpha_f(X, X), H \rangle - \frac{1}{n-1}\sum_{i=1}^{n} \|\alpha_f(X, X_i)\|^2$$

holds.

So, if $H(x) = 0$, one has $\alpha_f(X, X_i) = 0$ $(i = 1, \ldots, n)$. In particular, $\alpha_f(X, X) = 0$, which implies that $x \in M_{\rm tot}$, since X is arbitrary. Otherwise, $\langle \alpha_f(X, X), H \rangle$ is nonnegative. Thus, again from the fact that we are in codimension one, we have that α_f is semi-definite. This, together with Jonker's Theorem (for $c = 0$) and Theorem 1 (for $c \neq 0$), yields (i) and finishes our proof. □

In conclusion, we draw attention to the fact that the hypothesis on $M - M_{\rm tot}$ in Theorem 4 is necessary. To see this, let Σ_1 , Σ_2 be two $(n + 1)$-dimensional affine subspaces of $\mathbb{R}^{n+2}$ which intersect transversely. Set $\Sigma = \Sigma_1 \cup \Sigma_2$, $\Pi = \Sigma_1 \cap \Sigma_2$, and consider a compact isometric immersion $\bar{f} : M^n \to \mathbb{R}^{n+2}$ as shown in Figure 2; that is,

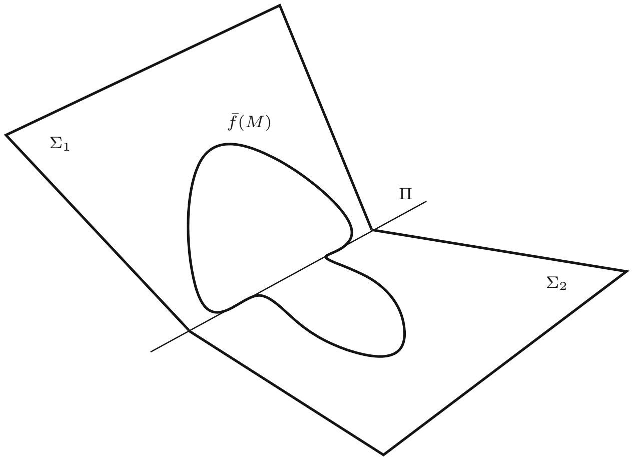

Fig. 2 An example showing the necessity of the hypothesis on $M - M_{\text{tot}}$ in Theorem 4

$\bar{f}(M) \subset \Sigma$, $\bar{f}(M) \cap (\Sigma_i - \Pi) \neq \emptyset$ $(i = 1, 2)$, and $\bar{f}(M_{\text{tot}})$ is contained in Π. Clearly, $\bar{f}$ is $(1\,;\,1)$-semi-regular, $M - M_{\text{tot}}$ is disconnected, and $\bar{f}$ does not admit any reduction of codimension. Hence, if $c \neq 0$, the same is true for $f = \varphi^{-1} \circ \bar{f} : M^n \to Q_c^{n+2}$, where φ is a suitable Beltrami map.

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