



## Translating solitons to flows by powers of the Gaussian curvature in Riemannian products

RONALDO FREIRE DE LIMA 

**Abstract.** We consider translating solitons to flows by positive powers  $\alpha$  of the Gaussian curvature, called  $K^\alpha$ -flows, in Riemannian products  $M \times \mathbb{R}$ . We prove that, when  $M$  is the Euclidean space  $\mathbb{R}^n$ , the sphere  $S^n$ , or one of the hyperbolic spaces  $\mathbb{H}_F^n$ , there exist complete rotational translating solitons to  $K^\alpha$ -flow in  $M \times \mathbb{R}$  for certain values of  $\alpha$ .

**Mathematics Subject Classification.** Primary 53B25; Secondary 53C42.

**Keywords.** Translating soliton, Gaussian curvature, Riemannian product.

**1. Introduction.** Over the last decades, the subject of extrinsic curvature flows in Riemannian manifolds has experienced a significant development. Along this time, special attention has been given to mean curvature and Gaussian curvature flows in Euclidean space, resulting in achievements such as the proof of short time existence of solutions and their convergence (after rescaling) to round spheres. (For a thorough account of these and other extrinsic flows, we refer to the recent book [5].)

Likewise, flows by powers of the Gaussian curvature, which we call  $K^\alpha$ -flows, have been considered by many authors, most notably B. Andrews (see, e.g., [2–4, 9, 12, 23, 29]). Among other reasons, the interest in  $K^\alpha$ -flows comes from the fact that they naturally relate to a wide range of research fields, varying from image processing to affine geometry and geometric analysis (cf. [1]).

In the context of extrinsic curvature flows in Euclidean space, there are special hypersurfaces, called *translating solitons*, which have the distinguished property of moving by translation under such a flow. A major feature of translating solitons is that they arise naturally as limits of rescalings near singularities of certain flows (cf. [6, 8, 18, 19]). Due to this and other facts, the construction and classification of complete translating solitons constitutes a major

problem in the theory of extrinsic geometric flows (see, e.g., [11, 13, 14, 25, 27–29] and the references therein).

There exists a special type of rotational strictly convex translating soliton in Euclidean space  $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ , called the *bowl soliton*, which is a graph over either the entire hyperplane  $\mathbb{R}^n \times \{0\}$  or an open ball in it. In the latter case, of course, the graph is cylindrically bounded. In [29], Urbas established the existence of entire bowl solitons to  $K^\alpha$ -flow in Euclidean space for all  $\alpha \in (0, 1/2]$ . A broad extension of this result was obtained by Rengaswami in [28], where he constructed and classified bowl solitons for a large class of extrinsic curvature flows. In addition, he established conditions on their velocity functions to determine whether such a soliton is entire or cylindrically bounded. For instance, bowl solitons to mean curvature flow are entire, whereas the ones to  $K^\alpha$ -flow are cylindrically bounded for  $\alpha = 1$  and entire for  $\alpha \in [0, 1/2)$ .

More generally, translating solitons can be defined in Riemannian products  $M \times \mathbb{R}$ , where  $M$  is an arbitrary Riemannian manifold (see Section 4). In [15], the author and G. Pipoli constructed and classified translating solitons (including bowl solitons) to higher order mean curvature flows in  $\mathbb{R}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$ . Results of the same nature were obtained in [24], where the authors considered translating solitons to mean curvature flow in  $M \times \mathbb{R}$ ,  $M$  being a Hadamard manifold with rotationally invariant metric (see also [21, 22]). We should mention as well the work of Pipoli [26], where he constructed and classified translating solitons to mean curvature flow in Heisenberg space.

In the present paper, we prove the existence of bowl solitons to  $K^\alpha$ -flow in  $M \times \mathbb{R}$  (for certain positive values of  $\alpha$ ) when  $M$  is the Euclidean space  $\mathbb{R}^n$ , the sphere  $\mathbb{S}^n$ , or one of the hyperbolic spaces  $\mathbb{H}_{\mathbb{F}}^m$  (rank-one symmetric spaces of non-compact type). In the Euclidean and hyperbolic cases, these solitons are entire, whereas in the spherical case, they project onto an open hemisphere  $B$  of  $\mathbb{S}^n$ , being asymptotic to the half-cylinder  $\partial B \times [0, +\infty)$ . In particular, we recover the aforementioned result of Urbas, giving it a new proof. We also mention that our results in Euclidean space are in accordance with [28, Theorems 1 and 2].

The paper is organized as follows. In Sect. 2, we fix notation regarding hypersurfaces of Riemannian products  $M \times \mathbb{R}$ . In Sect. 3, we discuss graphs defined on families of parallel hypersurfaces, setting some formulae. In Sect. 4, we introduce the  $K^\alpha$ -flow in  $M \times \mathbb{R}$  and establish a key lemma. Finally, in Sect. 5, we state and prove our main result.

**2. Preliminaries.** Given a Riemannian manifold  $M^n$ ,  $n \geq 2$ , consider the Riemannian product  $M \times \mathbb{R}$  endowed with its standard metric

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_M + dt^2,$$

and denote its Levi-Civita connection by  $\bar{\nabla}$ .

Let  $\Sigma^n$  be an oriented hypersurface of  $M \times \mathbb{R}$ . Set  $N$  for its unit normal field and  $A$  for its shape operator with respect to  $N$ , so that

$$AX = -\bar{\nabla}_X N, \quad X \in T\Sigma,$$

where  $T\Sigma$  stands for the tangent bundle of  $\Sigma$ . The principal curvatures of  $\Sigma$ , that is, the eigenvalues of the shape operator  $A$ , will be denoted by  $k_1, \dots, k_n$ . In this setting, the *Gaussian curvature* of  $\Sigma$  is the function  $K : \Sigma \rightarrow \mathbb{R}$  defined as

$$K := \det A = k_1 \cdots k_n.$$

We denote by  $\partial_t$  the gradient of the projection  $\pi_{\mathbb{R}}$  of  $M \times \mathbb{R}$  on the factor  $\mathbb{R}$ . Given a hypersurface  $\Sigma \subset M \times \mathbb{R}$ , its *height function*  $\xi$  and its *angle function*  $\Theta$  are defined by the following identities:

$$\xi := \pi_{\mathbb{R}}|_{\Sigma} \quad \text{and} \quad \Theta(x) := \langle N(x), \partial_t \rangle, \quad x \in \Sigma.$$

We shall denote the gradient of  $\xi$  on  $\Sigma$  by  $T$ , so that

$$T = \partial_t - \Theta N. \tag{1}$$

Given  $t \in \mathbb{R}$ , the set  $P_t := M \times \{t\}$  is called a *horizontal hyperplane* of  $M \times \mathbb{R}$ . Horizontal hyperplanes are all isometric to  $M$  and totally geodesic in  $M \times \mathbb{R}$ . In this context, we call a transversal intersection  $\Sigma_t := \Sigma \bar{\cap} P_t$  a *horizontal section* of  $\Sigma$ . Any horizontal section  $\Sigma_t$  is a hypersurface of  $P_t$ . So, at any point  $x \in \Sigma_t \subset \Sigma$ , the tangent space  $T_x \Sigma$  of  $\Sigma$  at  $x$  splits as the orthogonal sum

$$T_x \Sigma = T_x \Sigma_t \oplus \text{Span}\{T\}. \tag{2}$$

**Definition 1.** A hypersurface  $\Sigma \subset M \times \mathbb{R}$  is called *rotational* if there exists a fixed point  $o \in M$  such that any connected component of any horizontal section  $\Sigma_t$  of  $\Sigma$  is contained in a geodesic sphere of  $M \times \{t\}$  with center at  $o \times \{t\}$ . If so, the set  $\{o\} \times \mathbb{R}$  is called the *axis* of  $\Sigma$ .

Recall that the rank-one symmetric spaces of non-compact type are the hyperbolic spaces  $\mathbb{H}_{\mathbb{R}}^m$ ,  $\mathbb{H}_{\mathbb{C}}^m$ ,  $\mathbb{H}_{\mathbb{K}}^m$ , and  $\mathbb{H}_{\mathbb{O}}^2$ ,  $m \geq 1$ , called *real hyperbolic space*, *complex hyperbolic space*, *quaternionic hyperbolic space*, and *Cayley hyperbolic plane*, respectively. We will adopt the unified notation  $\mathbb{H}_{\mathbb{F}}^m$  for these spaces with  $m = 2$  for  $\mathbb{F} = \mathbb{O}$ . The real dimension of  $\mathbb{H}_{\mathbb{F}}^m$  is  $n = m \dim \mathbb{F}$ .

We remark that the hyperbolic spaces  $\mathbb{H}_{\mathbb{F}}^m$  are all Einstein–Hadamard manifolds whose sectional curvatures are negative and pinched. In particular, their geodesic spheres are all strictly convex (see, e.g., [7]).

**3. Graphs on parallel hypersurfaces.** Consider an isometric immersion

$$f : M_0^{n-1} \rightarrow M^n$$

between two Riemannian manifolds  $M_0^{n-1}$  and  $M^n$ , and suppose that there is a neighborhood  $\mathcal{U}$  of  $M_0$  in  $TM_0^{\perp}$  without focal points of  $f$ , that is, the restriction of the normal exponential map  $\exp_{M_0}^{\perp} : TM_0^{\perp} \rightarrow M$  to  $\mathcal{U}$  is a diffeomorphism onto its image. In this case, denoting by  $\eta$  the unit normal field of  $f$ , there is an open interval  $I \ni 0$  such that, for all  $p \in M_0$ , the curve

$$\gamma_p(s) = \exp_M(f(p), s\eta(p)), \quad s \in I, \tag{3}$$

is a well defined geodesic of  $M$  without conjugate points. Thus, for all  $s \in I$ ,

$$\begin{aligned} f_s &: M_0 \rightarrow M, \\ p &\mapsto \gamma_p(s), \end{aligned}$$

is an immersion of  $M_0$  into  $M$ , which is said to be *parallel* to  $f$ . Observe that, given  $p \in M_0$ , the tangent space  $f_{s*}(T_p M_0)$  of  $f_s$  at  $p$  is the parallel transport of  $f_*(T_p M_0)$  along  $\gamma_p$  from 0 to  $s$ . We also remark that, with the induced metric, the unit normal  $\eta_s$  of  $f_s$  at  $p$  is given by

$$\eta_s(p) = \gamma'_p(s).$$

**Definition 2.** Let  $\phi : I \rightarrow \phi(I) \subset \mathbb{R}$  be an increasing diffeomorphism, i.e.,  $\phi' > 0$ . With the above notation, we call the set

$$\Sigma := \{(f_s(p), \phi(s)) \in M \times \mathbb{R}; p \in M_0, s \in I\} \tag{4}$$

the *graph* determined by  $\{f_s; s \in I\}$  and  $\phi$ , or  $(f_s, \phi)$ -*graph* for short.

For an arbitrary point  $x = (f_s(p), \phi(s))$  of an  $(f_s, \phi)$ -graph  $\Sigma$ , one has

$$T_x \Sigma = f_{s*}(T_p M_0) \oplus \text{Span} \{\partial_s\}, \quad \partial_s = \eta_s + \phi'(s)\partial_t.$$

So, a unit normal to  $\Sigma$  is

$$N = \frac{-\phi'}{\sqrt{1 + (\phi')^2}}\eta_s + \frac{1}{\sqrt{1 + (\phi')^2}}\partial_t. \tag{5}$$

In particular, its angle function is

$$\Theta = \frac{1}{\sqrt{1 + (\phi')^2}}. \tag{6}$$

As shown in [16, Theorem 6]), any  $(f_s, \phi)$ -graph  $\Sigma$  has the *T-property*, meaning that  $T$  is a principal direction at any point of  $\Sigma$ . More precisely,

$$AT = \frac{\phi''}{(\sqrt{1 + (\phi')^2})^3}T. \tag{7}$$

Conversely, any hypersurface of  $M \times \mathbb{R}$  with non-vanishing angle function having  $T$  as a principal direction is given locally as an  $(f_s, \phi)$ -graph.

Given an  $(f_s, \phi)$ -graph  $\Sigma$ , let  $\{X_1, \dots, X_n\}$  be the orthonormal frame of principal directions of  $\Sigma$  in which  $X_n = T/\|T\|$ . In this case, for  $1 \leq i \leq n - 1$ , the fields  $X_i$  are all horizontal, that is, tangent to  $M$  (cf. (2)). Therefore, setting

$$\varrho := \frac{\phi'}{\sqrt{1 + (\phi')^2}} \tag{8}$$

and considering (5), we have, for all  $i = 1, \dots, n - 1$ , that

$$k_i = \langle AX_i, X_i \rangle = -\langle \bar{\nabla}_{X_i} N, X_i \rangle = \varrho \langle \bar{\nabla}_{X_i} \eta_s, X_i \rangle = -\varrho k_i^s,$$

where  $k_i^s$  is the  $i$ -th principal curvature of  $f_s$ . Also, it follows from (7) that  $k_n = \varrho'$ . Thus, the principal curvatures of the  $(f_s, \phi)$ -graph  $\Sigma$  at  $(f_s(p), \phi(s)) \in \Sigma$  are

$$k_i = -\varrho(s)k_i^s(p) \quad (1 \leq i \leq n - 1) \quad \text{and} \quad k_n = \varrho'(s). \tag{9}$$

In particular, the Gaussian curvature  $K$  of  $\Sigma$  at  $(f_s(p), \phi(s))$  is given by

$$K = (-\varrho(s))^{n-1} K_s(p) \varrho'(s), \tag{10}$$

where  $K_s$  denotes the Gaussian curvature of the hypersurface  $f_s$ .

We remark that the function  $\varrho$  defined in (8) determines the function  $\phi$ . Indeed, it follows from equality (8) that

$$\phi(s) = \int_{s_0}^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du + \phi(s_0), \quad s_0, s \in I. \tag{11}$$

In addition, from (6) and (8), the unit normal  $N$  defined in (5) can be written as  $N = -\varrho\eta_s + \Theta\partial_t$ . Hence, the relation

$$\Theta = \sqrt{1 - \varrho^2} \tag{12}$$

holds everywhere on any  $(f_s, \phi)$ -graph  $\Sigma$ . In particular, one has  $\varrho = \|T\|$ .

Now, let us assume that the family of parallel hypersurfaces

$$\mathcal{F} := \{f_s : M_0 \rightarrow M ; s \in I\}$$

is *isoparametric*, that is, for each  $s \in I$ , any principal curvature  $k_i^s$  of  $f_s \in \mathcal{F}$  is constant (possibly depending on  $i$  and  $s$ ). We shall assume further that each  $f_s$  is strictly convex with  $k_i^s < 0$ .

In this setting, the Gaussian curvature  $K_s$  of each hypersurface  $f_s$  is a (non-vanishing) function of  $s$  alone. Hence, from (10), the same is true for any  $(f_s, \phi)$ -graph  $\Sigma$  built on  $\mathcal{F}$ , that is,

$$K(f_s(p), \phi(s)) = (-\varrho(s))^{n-1} \varrho'(s) K_s, \quad (f_s(p), \phi(s)) \in \Sigma. \tag{13}$$

It should also be noticed that, by equalities (9), such a graph  $\Sigma$  is strictly convex.

**4. Translating  $K^\alpha$ -solitons in  $M \times \mathbb{R}$ .** Given a real number  $\alpha \neq 0$ , we say that an oriented strictly convex hypersurface  $\Sigma$  of  $M \times \mathbb{R}$  *moves under  $K^\alpha$ -flow* if there exists a one-parameter family of immersions  $F : M_0 \times [0, u_0) \rightarrow M \times \mathbb{R}$ ,  $u_0 \leq +\infty$ , such that

$$\begin{cases} \frac{\partial F^\perp}{\partial u}(p, u) = K^\alpha(p, u)N(p, u), \\ F(M_0, 0) = \Sigma, \end{cases} \tag{14}$$

where  $N(p, u)$  is the inward unit normal to the hypersurface  $F_u := F(\cdot, u)$ ,  $K(p, u)$  is the Gaussian curvature of  $F_u$  with respect to  $N_u := N(\cdot, u)$ , and  $\frac{\partial F^\perp}{\partial u}$  denotes the normal component of  $\frac{\partial F}{\partial u}$ , i.e.,

$$\frac{\partial F^\perp}{\partial u} = \left\langle \frac{\partial F}{\partial u}, N_u \right\rangle N_u.$$

In particular, the first equality in (14) is equivalent to

$$\left\langle \frac{\partial F}{\partial u}(p, u), N(p, u) \right\rangle = K^\alpha(p, u). \tag{15}$$

We call such a map  $F$  a  $K^\alpha$ -flow in  $M \times \mathbb{R}$ .

Denote by  $\exp$  the exponential map of  $M \times \mathbb{R}$  and consider an isometric immersion  $F_0 : M_0 \rightarrow M \times \mathbb{R}$ . Define then the map

$$F(p, u) := \exp_{F_0(p)}(u\partial_t), \quad (p, u) \in M_0 \times [0, +\infty),$$

and notice that, for each  $u \in (0, +\infty)$ , the hypersurface  $F(M_0, u)$  is nothing but a vertical translation of  $\Sigma := F(M_0, 0)$ . Since vertical translations are isometries of  $M \times \mathbb{R}$  keeping the second factor invariant, we have that  $\Sigma$  and  $F(M_0, u)$  are congruent with coinciding angle functions and Gaussian curvature, that is,

$$\Theta(p, u) = \Theta(p, 0) \quad \text{and} \quad K(p, u) = K(p, 0) \quad \forall (p, u) \in M_0 \times [0, u_0]. \quad (16)$$

Now, differentiating  $F$  with respect to  $u$ , we have

$$\frac{\partial F}{\partial u}(p, u) = (d \exp_{F_0(p)})(u \partial_t) \partial_t = \partial_t. \quad (17)$$

From (16) and (17), we have that  $F$  satisfies (15) if and only if the equality

$$\Theta(p, 0) = K^\alpha(p, 0)$$

holds on  $M_0$ . This fact motivates the following concept.

**Definition 3.** Given  $\alpha > 0$ , we say that an oriented strictly convex hypersurface  $\Sigma$  of the Riemannian product  $M \times \mathbb{R}$  is a *translating soliton to the  $K^\alpha$ -curvature flow* (or simply a *translating  $K^\alpha$ -soliton*) if the equality

$$K^\alpha = \Theta$$

holds everywhere on  $\Sigma$ . (From the above discussion, any such hypersurface  $\Sigma$  is the initial data of a  $K^\alpha$ -flow by vertical translations.)

Let us consider now an  $(f_s, \phi)$ -graph  $\Sigma$  in  $M \times \mathbb{R}$  such that the family

$$\mathcal{F} := \{f_s : M_0 \rightarrow M ; s \in I\}$$

is isoparametric with  $k_i^s < 0$ . In this case, it follows from (12) and (13) that  $\Sigma$  is a translating  $K^\alpha$ -soliton if and only if its associated  $\varrho$  function satisfies

$$(-\varrho(s))^{n-1} \varrho'(s) K_s = (1 - \varrho^2(s))^{\frac{1}{2\alpha}}, \quad s \in I. \quad (18)$$

Since  $K_s$  never vanishes on  $I$ , we have that (18) holds for  $\varrho$  if and only if

$$\tau'(s) = \frac{(-1)^{n-1} n (1 - \tau_n^{\frac{2}{n}})^{\frac{1}{2\alpha}}}{K_s}, \quad \tau := \varrho^n. \quad (19)$$

Summarizing, we have the following result.

**Lemma 1.** *Let  $\Sigma$  be an  $(f_s, \phi)$ -graph in  $M \times \mathbb{R}$  whose associated family*

$$\mathcal{F} := \{f_s : M_0 \rightarrow M ; s \in I\}$$

*of oriented parallel hypersurfaces is isoparametric with each of them having negative principal curvatures. Then,  $\Sigma$  is a translating  $K^\alpha$ -soliton if and only if (19) holds for  $\tau := \varrho^n$ , where  $\varrho : I \rightarrow \mathbb{R}$  is as in (8).*

**5. The main result.** For  $n \geq 2$  and  $m \geq 1$ , let  $M$  be one of the following manifolds, each of them endowed with its canonical Riemannian metric: Euclidean space  $\mathbb{R}^n$ , the sphere  $\mathbb{S}^n$ , or one of the hyperbolic spaces  $\mathbb{H}_{\mathbb{F}}^m$ . Then define

$$\mathcal{R}_M := \begin{cases} +\infty & \text{if } M \neq \mathbb{S}^n, \\ \pi/2 & \text{if } M = \mathbb{S}^n, \end{cases} \tag{20}$$

and consider a family

$$\mathcal{F} := \{f_s : \mathbb{S}^{n-1} \rightarrow M; s \in (0, \mathcal{R}_M)\} \tag{21}$$

of concentric geodesic spheres of  $M$  indexed by their radii, that is, for a fixed point  $o \in M$ , and for each  $s \in (0, \mathcal{R}_M)$ ,  $f_s(\mathbb{S}^{n-1})$  is the (strictly convex) geodesic sphere  $S_s(o)$  of  $M$  with center at  $o$  and radius  $s$ . In accordance to the notation of Section 3, for each  $s \in (0, \mathcal{R}_M)$ , we choose the outward orientation of  $f_s$ , so that *any principal curvature  $k_i^s$  of  $f_s$  is negative*.

For  $M = \mathbb{H}_{\mathbb{F}}^m$ , the principal curvatures  $k_i^s$  of the geodesic spheres  $f_s \in \mathcal{F}$  are:

$$k_1^s = -\frac{1}{2} \coth(s/2) \text{ with multiplicity } n - p - 1. \tag{22}$$

$$k_2^s = -\coth(s) \text{ with multiplicity } p,$$

where  $n = \dim \mathbb{H}_{\mathbb{F}}^m$ ,  $p = n - 1$  for  $\mathbb{H}^n$ ,  $p = 1$  for  $\mathbb{H}_{\mathbb{C}}^m$ ,  $p = 3$  for  $\mathbb{H}_{\mathbb{K}}^m$ , and  $p = 7$  for  $\mathbb{H}_{\mathbb{O}}^2$  (see, e.g., [10, pp. 353, 543] and [20]). Thus, the Gaussian curvature  $K_s$  of the geodesic sphere  $S_s(o)$  of  $\mathbb{H}_{\mathbb{F}}^m$  is given by

$$K_s = (-1)^{n-1} \left(\frac{1}{2} \coth(s/2)\right)^{n-p-1} (\coth s)^p.$$

In particular, the function

$$\frac{(-1)^{n-1}}{K_s} = (2 \tanh(s/2))^{n-p-1} (\tanh s)^p$$

is well defined and non-negative on  $[0, +\infty)$ .

As is well known, in the cases  $M = \mathbb{R}^n$  and  $M = \mathbb{S}^n$  one has

$$\frac{(-1)^{n-1}}{K_s} = s^{n-1} \quad \text{and} \quad \frac{(-1)^{n-1}}{K_s} = (\tan s)^{n-1},$$

respectively.

It is easily seen that, in any of these three cases, the equality

$$\lim_{s \rightarrow \mathcal{R}_M} \int_0^s \frac{(-1)^{n-1}}{K_v} dv = +\infty \tag{23}$$

holds. Finally, set

- $\delta_n := \begin{cases} 1/2 & \text{if } n = 2, \\ \max\{\frac{1}{4}, \frac{1}{n-1}\} & \text{if } n > 2. \end{cases}$
- $\mathcal{I}_M := \begin{cases} (0, 1/2] & \text{if } M \neq \mathbb{S}^n, \\ [\delta_n, 1/2] & \text{if } M = \mathbb{S}^n. \end{cases}$

With this notation, we now state and prove our main result.

**Theorem 1.** *Let  $M$  and  $\mathcal{I}_M \subset (0, 1/2]$  be as above. Then, for all  $\alpha \in \mathcal{I}_M$ , there exists a complete rotational strictly convex translating  $K^\alpha$ -soliton in the closed half-space  $M \times [0, +\infty)$  whose height function is unbounded. In addition, the following assertions hold:*

- For  $M \neq \mathbb{S}^n$ ,  $\Sigma$  is an entire graph over  $M$ .
- For  $M = \mathbb{S}^n$ ,  $\Sigma$  is a graph over an open hemisphere  $B \subset \mathbb{S}^n$ , being asymptotic to the half-cylinder  $\partial B \times [0, +\infty)$ .

*Proof.* Let  $\mathcal{F}$  be a family of parallel geodesic spheres of  $M$  as in (21). We intend to determine a function  $\phi \in C^\infty[0, \mathcal{R}_M)$  such that the corresponding  $(f_s, \phi)$ -graph  $\Sigma$  becomes the desired  $K^\alpha$ -soliton. With this purpose, let us consider the equality (19) as an ODE with variable  $\tau$ , and rewrite it as

$$\frac{d\tau}{(1 - \tau^{\frac{2}{n}})^{\frac{1}{2\alpha}}} = \frac{(-1)^{n-1}n}{K_s} ds. \tag{24}$$

In order to solve (24), consider the functions

$$\Phi: [0, 1) \rightarrow [0, +\infty) \quad \text{and} \quad \Psi: [0, \mathcal{R}_M) \rightarrow [0, +\infty)$$

given by

$$\Phi(\tau) := \int_0^\tau \frac{du}{(1 - u^{\frac{2}{n}})^{\frac{1}{2\alpha}}} \quad \text{and} \quad \Psi(s) := \int_0^s \frac{(-1)^{n-1}n}{K_v} dv.$$

It follows from (23) that  $\Psi$  is a diffeomorphism. Let us show that the same is true for  $\Phi$ . Indeed,  $\Phi$  is a  $C^\infty$  function with positive derivative, which implies that it is a diffeomorphism over its image. So, it remains to prove that  $\Phi([0, 1)) = [0, +\infty)$ . To this end, notice first that  $1 - u^{2/n} \leq 1 - u \forall u \in [0, 1)$ . Therefore, setting  $p := 1/(2\alpha) \geq 1$ , one has

$$\Phi(\tau) \geq \int_0^\tau \frac{du}{(1 - u)^p} = \begin{cases} \log\left(\frac{1}{1-\tau}\right) & \text{if } p = 1, \\ \frac{(1-\tau)^{1-p}-1}{p-1} & \text{if } p > 1, \end{cases}$$

which implies that  $\Phi(\tau) \rightarrow +\infty$  as  $\tau \rightarrow 1$ . Hence,  $\Phi([0, 1)) = [0, +\infty)$ , as asserted.

Now, we can define  $\tau : [0, \mathcal{R}_M) \rightarrow \mathbb{R}$  by

$$\tau(s) = \Phi^{-1}(\Psi(s)), \tag{25}$$

which is clearly a solution of (24) (and so of (19)) satisfying

$$0 = \tau(0) \leq \tau(s) < 1 \quad \forall s \in [0, \mathcal{R}_M) \quad \text{and} \quad \lim_{s \rightarrow \mathcal{R}_M} \tau(s) = 1. \tag{26}$$

Therefore, by Lemma 1, the corresponding  $(f_s, \phi)$ -graph  $\Sigma$  is a translating  $K^\alpha$ -soliton in  $M \times \mathbb{R}$ . Moreover, since  $\rho = \tau^{1/n}$ , we have from (11) (with  $s_0 = 0$ ) and (26) that the function  $\phi$  is defined in  $[0, \mathcal{R}_M)$  and satisfies

$$\phi(0) = \phi'(0) = 0 \quad \text{and} \quad \lim_{s \rightarrow \mathcal{R}_M} \phi'(s) = +\infty.$$

We conclude from the above discussion that, if  $M$  is either  $\mathbb{R}^n$  or  $\mathbb{H}_{\mathbb{F}}^m$ , then  $\Sigma$  is an unbounded entire graph over  $M$  which is contained in the closed half-space  $M \times [0, +\infty)$ . In particular,  $\Sigma$  is complete (Fig. 1).

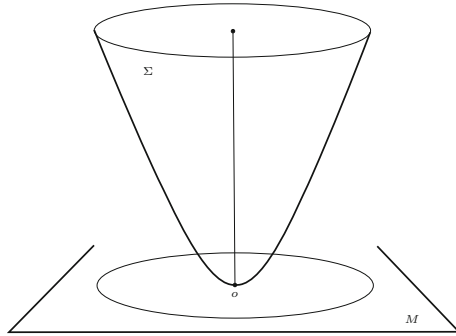


FIGURE 1. A translating  $K^\alpha$ -soliton in  $M \times \mathbb{R}$

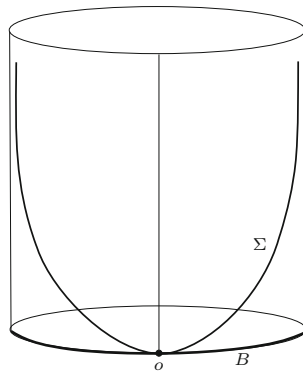


FIGURE 2. A translating  $K^\alpha$ -soliton in  $\mathbb{S}^n \times \mathbb{R}$

If  $M = \mathbb{S}^n$ , then  $\Sigma$  is a graph over the open hemisphere  $B$  centered at  $o \in \mathbb{S}^n$  (Fig. 2). Thus, to conclude that  $\Sigma$  is complete and asymptotic to  $\partial B \times [0, +\infty)$ , we must prove that  $\phi$  is unbounded.

For  $n = 2$ , we have from the hypothesis that  $\alpha = 1/2$ . In this case, it is easily checked that  $\varrho(s) = \sin s$  is the solution of (18) satisfying  $\varrho(0) = 0$ . So, from (11),

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du = \int_0^s \tan(u) du = \log \left( \frac{1}{\cos s} \right), \quad s \in (0, \pi/2),$$

which implies that  $\phi(s) \rightarrow +\infty$  as  $s \rightarrow \pi/2$ .

For  $n > 2$ , let us show first that  $\tau'$  is bounded in  $(0, \pi/2)$ . Indeed, from (19),

$$\tau'(s) = n(\tan s)^{n-1}(1 - (\tau(s))^{2/n})^p, \quad p = \frac{1}{2\alpha}.$$

So, setting

$$\mu_1(s) := n(\tan s)^{n-2}(\sec s)^2(1 - (\tau(s))^{2/n})^p \text{ and } \mu_2(s) := (1 - (\tau(s))^{2/n}) \tan s,$$

a straightforward calculation yields

$$\begin{aligned} \tau''(s) &= \mu_1(s) \left[ (n-1) - \frac{2p(\tau(s))^{\frac{2-n}{n}} (\cos s)^2 (\tan s) \tau'(s)}{n(1 - (\tau(s))^{\frac{2}{n}})} \right] \\ &= \mu_1(s) \left[ (n-1) - \frac{2p(\tau(s))^{\frac{2-n}{n}} (\sin s)^2 \tau'(s)}{n\mu_2(s)} \right], \quad s \in (0, \pi/2). \end{aligned} \tag{27}$$

Let us suppose, by contradiction, that there exists a sequence  $(s_k)$  in  $(0, \pi/2)$  such that  $\tau'(s_k) \rightarrow +\infty$ . We can assume, without loss of generality, that  $\tau''(s_k) > 0$  for all  $k \in \mathbb{N}$ . Under this assumption, we have from (27) that the sequence  $\mu_2(s_k)$  is unbounded above. Otherwise, for a sufficiently large  $k$ , the expression in the brackets would be negative for  $s = s_k$ , which would give  $\tau''(s_k) < 0$ , a contradiction.

So, we can assume  $\mu_2(s_k) \rightarrow +\infty$ . However, from our choice of  $\alpha$ , we have that  $0 \leq p-1 \leq 1$ , which yields  $(1 - (\tau(s_k))^{2/n})^{p-1} \geq (1 - (\tau(s_k))^{2/n})$ . Thus,

$$\frac{\tau'(s_k)}{\mu_2(s_k)} = n(\tan s_k)^{n-2} (1 - (\tau(s_k))^{2/n})^{p-1} \geq n(\tan s_k)^{n-3} \mu_2(s_k) \rightarrow +\infty,$$

which, together with equality (27), gives  $\tau''(s_k) < 0$  for a sufficiently large  $k$ , again a contradiction. Therefore,  $\tau'$  (and so  $\varrho'$ ) is bounded in  $(0, \pi/2)$ .

Now, let  $C > 0$  be such that  $\varrho'(s) < C$  for all  $s \in (0, \pi/2)$ . Since we are assuming  $\alpha(n-1) \geq 1$ , equalities (11) and (18) yield

$$\phi(s) = \int_0^s \frac{\varrho(u)}{\sqrt{1 - \varrho^2(u)}} du = \int_0^s \frac{(\tan u)^{\alpha(n-1)}}{(\varrho(u))^{\alpha(n-1)-1} (\varrho'(u))^\alpha} du \geq \frac{1}{C^\alpha} \int_0^s \tan u \, du,$$

which implies that  $\phi(s) \rightarrow +\infty$  as  $s \rightarrow \pi/2$ . This concludes the proof. □

We finish by noting that, for a given Riemannian manifold  $M$ , we have from Lemma 1 that there exist local translating  $K^\alpha$ -solitons in  $M \times \mathbb{R}$  as long as  $M$  admits families of isoparametric, strictly convex hypersurfaces. This applies, for instance, to the hyperbolic spaces  $\mathbb{H}_{\mathbb{F}}^m$  and their families of parallel horospheres (see [7, Proposition-(vi)]), to the real hyperbolic space  $\mathbb{H}_{\mathbb{R}}^n$  and its families of equidistant hypersurfaces to a given totally geodesic hyperplane, as well as to  $\mathbb{S}^n$  and any of its many families of strictly convex isoparametric hypersurfaces (cf. [17] and the references therein).

**Acknowledgements.** The author is indebt to Isabel L. Rios and Roberto T. de Oliveira for helpful conversations during the preparation of this paper.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

## References

- [1] Andrews, B.: Motion of hypersurfaces by Gauss curvature. *Pacific J. Math.* **195**(1), 1–34 (2000)
- [2] Andrews, B., Chen, X.: Surfaces moving by powers of Gauss curvature. *Pure Appl. Math.* **8**(4), 825–834 (2012)
- [3] Andrews, B., Guan, P., Ni, L.: Flow by powers of the Gauss curvature. *Adv. Math.* **299**, 174–201 (2016)
- [4] Andrews, B., McCoy, J.: Convex hypersurfaces with pinched principal curvatures and flow of convex hypersurfaces by high powers of curvature. *Trans. Amer. Math. Soc.* **364**(7), 3427–3447 (2012)
- [5] Andrews, B., Chow, B., Guenther, C., Langford, M.: *Extrinsic Geometric Flows*. Graduate Studies in Mathematics, 206. American Mathematical Society, Providence, RI (2020)
- [6] Angenent, S.B., Velázquez, J.J.L.: Degenerate neckpinches in mean curvature flow. *J. Reine Angew. Math.* **482**, 15–66 (1997)
- [7] Berndt, J., Tricerri, F., Vanhecke, L.: *Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces*. Lecture Notes in Mathematics, 1598. Springer, Berlin (1995)
- [8] Bourni, T., Langford, M.: Type-II singularities of two-convex immersed mean curvature flow. *Geom. Flows* **2**(1), 1–17 (2017)
- [9] Brendle, S., Choi, K., Daskalopoulos, P.: Asymptotic behavior of flows by powers of the Gaussian curvature. *Acta Math.* **219**(1), 1–16 (2017)
- [10] Cecil, T., Ryan, P.: *Geometry of Hypersurfaces*. Springer, Berlin (2015)
- [11] Choi, B., Choi, K., Daskalopoulos, P.: Uniqueness of ancient solutions to Gauss curvature flow asymptotic to a cylinder. [arXiv:2004.11754](https://arxiv.org/abs/2004.11754) (2022)
- [12] Choi, K., Daskalopoulos, P., Kim, L., Lee, K.-A.: The evolution of complete non-compact graphs by powers of Gauss curvature. *J. Reine Angew. Math.* **757**, 131–158 (2019)
- [13] Choi, K., Daskalopoulos, P., Lee, K.-A.: Translating solutions to the Gauss curvature flow with flat sides. *Anal. PDE* **14**(2), 595–616 (2021)
- [14] Clutterbuck, J., Schnürer, O.C., Schulze, F.: Stability of translating solutions to mean curvature flow. *Calc. Var.* **29**, 281–293 (2007)
- [15] de Lima, R.F., Pipoli, G.: Translators to higher order mean curvature flows in  $\mathbb{R}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$ . [arXiv:2211.03918](https://arxiv.org/abs/2211.03918) (2022)
- [16] de Lima, R.F., Roitman, P.: Helicoids and catenoids in  $M \times \mathbb{R}$ . *Ann. Mat. Pura Appl.* **200**, 2385–2421 (2021)
- [17] Domínguez-Vázquez, M.: An introduction to isoparametric foliations. Preprint (2018). <http://xtsunxet.usc.es/miguel/teaching/jae2018.html>
- [18] Hamilton, R.: Harnack estimate for the mean curvature flow. *J. Differ. Geom.* **41**(1), 215–226 (1995)
- [19] Huisken, G., Sinestrari, C.: Convexity estimates for mean curvature flow and singularities of mean convex surfaces. *Acta Math.* **183**(1), 45–70 (1999)

- [20] Kim, S., Nikolayevsky, Y., Park, J.: Einstein hypersurfaces of the Cayley projective plane. *Differ. Geom. Appl.* **69**, 1–6 (2020)
- [21] Kocakusakli, E., Ortega, M.: Extending translating solitons in semi-Riemannian manifolds. In: *Lorentzian Geometry and Related Topics*. Springer Proc. Math. Stat., 211. Springer, Cham (2017)
- [22] Lawn, M.A., Ortega, M.: Translating solitons from semi-Riemannian submersions. [arXiv:1607.04571](https://arxiv.org/abs/1607.04571) (2018)
- [23] Li, X., Wang, K.: Nonparametric hypersurfaces moving by powers of Gauss curvature. *Michigan Math. J.* **66**(4), 675–682 (2017)
- [24] Lira, J.H., Martín, F.: Translating solitons in Riemannian products. *J. Differential Equations* **266**, 7780–7812 (2019)
- [25] Martín, F., Savas-Halilaj, A., Smoczyk, K.: On the topology of translating solitons of the mean curvature flow. *Calc. Var. Partial Differential Equations* **54**(3), 2853–2882 (2015)
- [26] Pipoli, G.: Invariant translators of the Heisenberg group. *J. Geom. Anal.* **31**, 5219–5258 (2021)
- [27] Spruck, J., Xiao, L.: Complete translating solitons to the mean curvature flow in  $\mathbb{R}^3$  with nonnegative mean curvature. *Amer. J. Math.* **142**(3), 993–1015 (2020)
- [28] Rengaswami, S.: Rotationally symmetric translating solutions to extrinsic geometric flows. [arXiv:2109.10456v1](https://arxiv.org/abs/2109.10456v1) (2021)
- [29] Urbas, J.: Complete noncompact self-similar solutions of Gauss curvature flows. I. Positive powers. *Math. Ann.* **311**(2), 251–274 (1998)

RONALDO FREIRE DE LIMA  
Departamento de Matemática  
Universidade Federal do Rio Grande do Norte  
Natal  
Brazil  
e-mail: [ronaldo.freire@ufrn.br](mailto:ronaldo.freire@ufrn.br)

Received: 19 October 2021

Revised: 1 February 2023

Accepted: 11 February 2023